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Decision Support

Penalty functions based upon a general class of restricted dissimilarity functions

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ABSTRACT

In this paper the notion of restricted dissimilarity function is discussed and some general results are shown. The relation between the concepts of restricted dissimilarity function and penalty function is presented. A specific model of construction of penalty functions by means of a wide class of restricted dissimilarity functions based upon automorphisms of the unit interval is studied. A characterization theorem of the automorphisms which give rise to two-dimensional penalty functions is proposed. A generalization of the previous theorem to any dimension $n > 2$ is also provided. Finally, a not convex example of generator of penalty functions of arbitrary dimension is illustrated.

1. Introduction

The notion of *penalty based aggregation function* has become increasingly popular in the literature over the last few years (see, for instance, Calvo & Beliakov, 2010 and the references therein). In Bustince, Jurio, Pradera, Mesiar, and Beliakov (2013), the authors suggest the use of penalty functions for selecting alternatives in decision making problems. In particular, they build penalty functions by means of a particular class of *restricted dissimilarity functions* (see Bustince, Barrenechea, & Pagola, 2008), called *faithful restricted dissimilarity functions*, in order to generalize one of the most widely used methods in decision making, that is the weighted voting method (see, for example, Hüllermeier & Brinker, 2008; Hüllermeier & Vanderlooy, 2010). From a mere theoretical point of view, a faithful restricted dissimilarity function is strictly related to a convex automorphism of the real unit interval up to a bijection. As the authors somehow admit, there is no real reason for imposing convexity restriction other than to assure that the corresponding penalty function is convex, so fulfilling, *a fortiori*, the crucial property of quasi-convexity demanded to all penalty functions.

This consideration has led us to the main goal of this paper: to characterize a class of restricted dissimilarity functions, wider than the faithful restricted dissimilarity functions, able to assure that the generated one-variable mappings, constructed in the same way as in Bustince et al. (2013), turn out to be penalty functions.

We have organized this paper as follows: In the next section we introduce the basic notions needed for subsequent developments. In Section 3 we analyze the limited cases in which a restricted dissimilarity function is also a distance and mostly we study a model of construction of penalty functions by means of a general subclass of restricted dissimilarity functions strictly connected to automorphisms of the real unit interval, also called generators. In Section 4, we characterize the class of generators of two-dimensional penalty functions, while in Section 5 we extend the two-dimensional result to any dimension $n > 2$. Finally, we present a conclusion and some references.

2. Basic notions and first results

In this article, we will make use of the following notations and assumptions.

Our domain of interest is the real unit interval, denoted by $\mathbb{I}$, being clear that it might be replaced without loss of generality by any closed, non-empty subinterval of the real line. We adopt the classical notation $\mathbf{x} = (x_1, \ldots, x_n)$ for any n-tuple $\mathbf{x}$ in $\mathbb{I}^n$, while $\mathbf{x}_{\nearrow} = (x_{(1)}, \ldots, x_{(n)})$ represents the result of the permutation of the components of $\mathbf{x}$ in increasing order, i.e. $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$. We denote by W_n the set of *weighting vectors* of dimension n, i.e. $W_n = \{\mathbf{w} \in]0, 1[^n : \sum_{i=1}^n w_i = 1\}$. We will exclusively reserve the symbol $\mathbf{w}$ for any weighting vector: further, $\mathbf{w}_n^*$ stands for the weighting vector given by $(1/n, \ldots, 1/n)$.

Definition 2.1. Let $x_0 \in \mathbb{I}$, $\mu > 0$ and $E \subset \mathbb{I}$. Then, we set

$$\mu E = \{\mu \cdot x : x \in E\} \tag{2.1}$$

and

$$E \pm \{x_0\} = \{t \pm x_0 : t \in E\}. \tag{2.2}$$

We warn the reader that throughout the paper the notion of monotonicity is intended in weak sense: otherwise, we speak of strict monotonicity. Moreover, when we say that a property holds almost

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everywhere (for short, a.e.), it is intended that it is true outside a set of Lebesgue measure, denoted by λ, equal to zero.

Consider now the following definition.

Definition 2.2 (Bustince, Fernandez, Mesiar, Pradera, & Beliakov, 2011; Calvo & Beliakov, 2010)**.** Let $P : \mathbb{I}^{n+1} \to [0, \infty]$. We say that P is a penalty function of dimension n if, and only if, satisfies:

(i) $P(\mathbf{x}, y) = 0$ if $x_i = y$ for all $i \in \{1, \ldots, n\}$;
(ii) for every fixed $\mathbf{x} \in \mathbb{I}^n$, the set of minimizers of the mapping $y \mapsto P(\mathbf{x}, y)$ is a subinterval of $\mathbb{I}$, possibly reducible to a singleton.

The penalty based function $f : \mathbb{I}^n \to \mathbb{I}$ is given by

$$f(\mathbf{x}) = \arg\min_y P(\mathbf{x}, y)$$

if y is the unique minimizer or $y = (a + b)/2$ if the set of minimizers is a subinterval of $\mathbb{I}$ with bounds a and b.

A penalty function is a tool for measuring the disagreement between the input $\mathbf{x}$ and the value y. The penalty based function f associates with any input $\mathbf{x}$ the corresponding output value y which just minimizes the chosen disagreement.

The first, prototypical model of penalty function appeared in the literature was of the form

$$P(\mathbf{x}, y) = \sum_{i=1}^{n} d_p(x_i, y), \tag{2.3}$$

where $d_p : \mathbb{I}^2 \to \mathbb{I}$ is given by $d_p(x, y) = |x - y|^p$, for $p \geq 1$. In particular, the cases $p = 1$ and $p = 2$ were already studied by Fermat, Laplace and Cauchy (see Hudry, Leclerc, Monjardet, & Barthélemy, 2010; Torra & Narukawa, 2007 and the references therein). When $p = 1$, the penalty function is obtained as sum of the (Euclidean) distances of the components of the input $\mathbf{x}$ to the value y. The model is then generalized replacing d_1 with d_p, with the crucial difference that only d_1 is a distance in mathematical sense. We will return to this point later.

The notion of penalty function is closely related to the dissimilarity function, as proposed and discussed in Mesiar (2007), even if this concept in its turn is inspired to a first formulation of penalty function given by Calvo, Mesiar, and Yager (2004) which is quite different from the above one. Our version fundamentally coincides with the most general one as it appears in Calvo and Beliakov (2010), except for the redundant requirement $P(\mathbf{x}, y) \geq 0$ for all $\mathbf{x}, y$, here dropped.

In some papers, condition (ii) is replaced by the requirement that the mapping $y \mapsto P(\mathbf{x}, y)$ is quasi-convex for any fixed $\mathbf{x} \in \mathbb{I}^n$. Recall that a real function g over a convex subset X of $\mathbb{R}^n$ is quasi-convex if its level sets $L_c = \{\mathbf{x} \in X : g(\mathbf{x}) \leq c\}$ are convex (see Greenberg & Pierskalla, 1971 for a thorough exposition of the notion of quasi-convexity). It is well-known that if g is a function of a single variable, then g is quasi-convex if, and only if, either it is monotone or there exists a $x^* \in X$ such that g is decreasing on $\{x \in X : x \leq x^*\}$ and increasing on $\{x \in X : x \geq x^*\}$. Therefore, it is clear that quasi-convexity is a stronger condition than (ii).

The restricted dissimilarity functions were introduced by Bustince et al. (2008) and are inspired, among others, to the notions of proximity and dissimilarity measures, as they appear in Fan and Xie (1999) and Liu (1992), respectively. In the applications, they turn out to be very useful, for instance, in image processing, in order to measure the dissimilarity of two objects, while, from a theoretical point of view, they are more flexible than existing dissimilarity functions.

Definition 2.3. A mapping $d_R : \mathbb{I}^2 \to \mathbb{I}$ is called a restricted dissimilarity function if :

(D1) $d_R(x, y) = d_R(y, x)$ for every $x, y \in \mathbb{I}$;
(D2) $d_R(x, y) = 1$ if, and only if, $\{x, y\} = \{0, 1\}$;
(D3) $d_R(x, y) = 0$ if, and only if, $x = y$;
(D4) $d_R(y, z) \leq d_R(x, t)$ for all $x, y, z, t \in \mathbb{I}$ such that $x \leq y \leq z \leq t$.

In the sequel, we will exclusively deal with continuous restricted dissimilarity functions. This is not a too strong assumption, since we know that, fixed any $x \in \mathbb{I}$, the mapping $t \mapsto d_R(x, t)$ is quasi-convex (see Bustince et al., 2011), hence it is also continuous on $\mathbb{I}$ up to a subset E_x of $\mathbb{I}$ such that $\lambda(E_x) = 0$ (see Greenberg & Pierskalla, 1971). The next result just shows that if $E_x = \emptyset$ for all $x \in \mathbb{I}$, then d_R is continuous as two-place function.

Proposition 2.4. *Let d_R be a restricted dissimilarity function. Then d_R is continuous if, and only if, the mapping $t \mapsto d_R(x, t)$ is continuous on $\mathbb{I}$ for every fixed $x \in \mathbb{I}$.*

Proof. First of all, notice that, by (D1), the assumption may be equivalently formulated as continuity of the mapping $t \mapsto d_R(t, x)$ for every fixed $x \in \mathbb{I}$. Given an arbitrary point (x_0, y_0) of $\mathbb{I}^2$, we have to show that for any real $\varepsilon > 0$ there exists a neighborhood U of (x_0, y_0) such that

$$d_R(x_0, y_0) - \varepsilon \leq d_R(x, y) \leq d_R(x_0, y_0) + \varepsilon$$

for all $(x, y) \in U$. Let us divide the proof into four cases, according to the position of (x_0, y_0).

Case (1): $0 < x_0 < y_0 < 1$. By continuity of $t \mapsto d_R(x_0, t)$ at $t = y_0$, we can always find a $\delta \in]0, \min\{x_0, 1 - y_0, (y_0 - x_0)/2\}[$ such that

$$|d_R(x_0, t) - d_R(x_0, y_0)| \leq \varepsilon/2 \tag{2.4}$$

for all $t \in [y_0 - \delta, y_0 + \delta]$. Moreover, by continuity of $t \mapsto d_R(t, y_0 + \delta)$ at $t = x_0$, we can always find a $\delta_1 \in]0, \delta[$ such that

$$d_R(x_0 - \delta_1, y_0 + \delta) \leq d_R(x_0, y_0 + \delta) + \varepsilon/2.$$

Employing Eq. (2.4), last inequality leads to

$$d_R(x_0 - \delta_1, y_0 + \delta) \leq d_R(x_0, y_0) + \varepsilon. \tag{2.5}$$

In the same way, starting with the continuity of the mapping $t \mapsto d_R(t, y_0 - \delta)$ at $t = x_0$, there exists a $\delta_2 \in]0, \delta[$ such that

$$d_R(x_0 + \delta_2, y_0 - \delta) \geq d_R(x_0, y_0) - \varepsilon. \tag{2.6}$$

Set $\delta^* := \min\{\delta_1, \delta_2\}$: by the properties of δ, it is quite easy to see that

$$x_0 - \delta_1 \leq x_0 - \delta^* < x_0 + \delta^* \leq x_0 + \delta_2 < y_0 - \delta,$$

hence, by (D4), one immediately finds

$$d_R(x_0 + \delta^*, y_0 - \delta) \leq d_R(x, y) \leq d_R(x_0 - \delta^*, y_0 + \delta)$$

for all $(x, y) \in [x_0 - \delta^*, x_0 + \delta^*] \times [y_0 - \delta, y_0 + \delta]$, and the claim directly follows from Eqs. (2.5) and (2.6).

Case (2): $0 < y_0 < x_0 < 1$. It directly follows from the previous case, taking into account (D1).

Case (3): $0 < x_0 = y_0 < 1$. The proof is the same (even simpler) as that of the first case.

Case (4): (x_0, y_0) belongs to the boundary of $\mathbb{I}^2$. This is again a sub-case of the first one, so concluding the proof. □

In what follows, AC($\mathbb{I}$) denotes the family of absolutely continuous real functions over $\mathbb{I}$. We say that any $\varphi : \mathbb{I} \to \mathbb{I}$ belongs to $\mathcal{A}(\mathbb{I})$ if, and only if,

(A1) $\varphi \in \mathrm{AC}(\mathbb{I})$;
(A2) φ is an increasing bijection.

Obviously, any $\varphi \in \mathcal{A}(\mathbb{I})$ is an automorphism of the real unit interval (see, for instance, Fodor & Roubens, 1994). Recall that, as a consequence of (A1), the derivative of any $\varphi \in \mathcal{A}(\mathbb{I})$ exists and is defined on $\mathbb{I}$ up to a subset of measure zero (see, for instance, Yeh, 2006).

Let us introduce now a special class of restricted dissimilarity functions by means of a construction illustrated in the next lemma, whose elementary proof is omitted. Later on, any continuous bijection $h : \mathbb{I} \to \mathbb{I}$ is simply called a *scaling function*.