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Decision Support

Extreme point-based multi-attribute decision analysis with incomplete information

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ABSTRACT

In this paper, we present a simple method for finding the extreme points of various types of incomplete attribute weights. Incomplete information about attribute weights is transformed by a sequence of change of variables to a set whose extreme points are readily found. This enhanced method fails to derive the extreme points of *every* type of incomplete attribute weights. Nevertheless, it provides us with a flexible method for finding the extreme points, including widely-used forms of incomplete attribute weights. Finally, incomplete attribute values, expressed in various forms, are also analyzed to find their characterizing extreme points by applying similar procedures carried out in the incomplete attribute weights.

1. Introduction

The multiple-attribute decision-making (MADM) problems have been studied extensively over the past several decades. In general, most of studies are based on precise parameters of the problems by one of elaborate elicitation methods. Intermittently, however, attempts emerged to alleviate the burdens of specifying parameters due to the reasons of time pressure, lack of data and domain knowledge, limited attention and information processing capabilities and so on. Of course, prior (sophisticated) methodologies are advocated if a decision-maker is willing to or able to supply, with the help of decision analyst, all the information necessary to solve the MADM problems at hand.

Many earlier studies on incomplete attribute weights are presented in the context of MADM (Ahn & Park, 2008; Eum, Park, & Kim, 2001; Kirkwood & Sarin, 1985; Lee, Park, & Kim, 2002; Mateos, Jiménez, & Ríos-Insua, 2006; Mateos, Jiménez, & Blanco, 2012; Mateos, Ríos-Insua, & Jiménez, 2007; Park, 2004; Puerto, Mármol, Monloy, & Fernandez, 2000). Examples of incomplete attribute weights are in the form of weak or strict rankings, difference rankings, and fixed or ratio bounds. The incomplete information about values could occur in practice as well. For a qualitative attribute, a decision-maker may say an alternative is best (100%), and another is in the level between 80% and 90% relative to the level of the first one (Ahn, Park, Han, & Kim, 2000; Kim & Ahn, 1999). This can be expressed in the form of ratio bounds.

To solve MADM problems whose elements are incompletely known, we rely on one of two approaches: a linear programming (LP) approach and an extreme point approach. The former attempts to formulate and then solve the problems which are constrained by incomplete attribute weights. The extreme point approach, on the other hand, tries to find the extreme points characterizing the incomplete attribute weights (Mármol, Puerto, & Fernandez, 1998; Mármol, Puerto, & Fernandez, 2002; Puerto et al., 2000; Solymosi & Dombi, 1986). Once they are found, dominance relations between alternatives are established by simply multiplying extreme points of incomplete attribute weights by performance evaluations of alternatives in the case that only the attribute weights are incomplete (see also Section 3 for the case of incomplete attribute weights and values). It seems that the extreme point approach can be better used to test dominance only if they are easily and completely identified. To this end, Puerto et al. (2000) presented some formulas to find the extreme points of widely-used incomplete attribute weights. For more details, please also refer to the papers (Mármol et al., 1998; Mármol et al., 2002; Puerto et al., 2000; Shepetukha & Olson, 2001).

In this paper, we present a simple method for finding the extreme points of incomplete attribute weights by the change of variables, which has some merits as follows. First of all, the proposed method is easy to apply but the results are equivalent to those of Puerto et al. (2000). The method rather encompasses a type of incomplete information such as multi-levels of preference

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differences (Section 2.5). Second, the proposed method is readily extendable to cover other types of incomplete attribute weights beyond the ones mentioned in the paper. An example is illustrated at the end of Section 2. Moreover, the change of variable technique can be effectively used to address bounded descriptions. Finally, incomplete attribute values, expressed in various forms, are also analyzed to find their characterizing extreme points by applying similar procedures carried out in the incomplete attribute weights. These findings complete a study on the extreme point-based MADM under incomplete information. A practical use of extreme points of strict preferences in Section 3.1, for example, is to represent judgments induced by the rank order. Thus we can analyze a qualitative multi-criteria decision-making problem where each alternative is assigned a rank position and multiple criteria are ordinally ranked in terms of relative importance in a slightly different context with Cook and Kress (1996).

The paper is comprised of two main sections. Sections 2 and 3 describe how to find the extreme points of incomplete attribute weights and values respectively. A numerical example is illustrated in Section 4, followed by concluding remarks in Section 5.

2. Identifying extreme points of incomplete attribute weights

In MADM problems, one usually considers a finite discrete set of alternatives $A = \{x_1, x_2, \ldots, x_m\}$, which is valued by a finite discrete set of attributes $C = \{c_1, c_2, \ldots, c_k\}$ where m is the number of alternatives and k the number of attributes. We assume an additive model, which is considered a valid approximation in most real decision-making problems (Stewart, 1996), and is widely used within multi-attribute value theory (MAVT) (Keeney & Raiffa, 1993; von Winterfeldt & Edwards, 1986),

$$V(x_i) = \sum_{j=1}^{k} \lambda_j v_j(x_{ij})$$

where $V(x_i)$ is the overall multi-attribute value of alternative x_i with attribute levels $(x_{i1}, \ldots, x_{ik}), 0 \leqslant V(x_i) \leqslant 1, i = 1, \ldots, m$; λ_j is a weighting coefficient such that $\sum_{j=1}^{k} \lambda_j = 1, \lambda_j \geqslant 0, j = 1, \ldots, k$; $v_j(\cdot)$ is a single attribute value of alternatives with respect to jth attribute, $j = 1, \ldots, k$, $0 \leqslant v_j(\cdot) \leqslant 1$.

Let Λ denote a set of incomplete attribute weights and V_j a set of outcomes on the jth attribute, that is, the values of $v_j(\cdot)$ for alternatives such that $V_j = \{v_j(x_{1j}), v_j(x_{2j}), \ldots, v_j(x_{mj})\}$. According to pairwise dominance, alternative x_p dominates x_q if for any fixed set of feasible weights the worst outcome in x_p at least exceeds the best outcome in x_q (Salo & Hämäläinen, 1992; Weber, 1987). Specifically, alternative x_p is at least preferred to x_q if and only if $PD(x_p, x_q) \geqslant 0$ where

$$PD(x_p, x_q) = \min\left\{ \sum_{j=1}^{k} \lambda_j \delta_j(x_p, x_q) \,\middle|\, \lambda_j \in \Lambda \right\}, \qquad (1)$$

$$\delta_j(x_p, x_q) = \min\left\{ v_j(x_{pj}) - v_j(x_{qj}) \,\middle|\, v_j \in V_j \right\}.$$

In the paper, we assume that the importance of attributes is specified by one of various types of incomplete attribute weights, which are listed in accordance with the development by Puerto et al. (2000): (a) lower bounds on the weighting coefficients, (b) rank ordered attributes, (c) ratio scale inequalities among attributes, and (d) rank ordered attributes with discriminating factors. In addition to these incomplete attribute weights, we append (e) multi-levels of preference differences,[1] which can be subsequently constructed based on the rank ordered attributes. See also the paper for extensive examples of incomplete attribute weights by Park (2004). Given these incomplete attribute weights, the proposed method attempts to find all extreme points for each case by the change of variables.

2.1. Lower bounds on the weighting coefficients (LB)

A set of lower bounds on the weighting coefficients is denoted by Λ_{LB}:

$$\Lambda_{LB} = \left\{ \boldsymbol{\lambda} : \lambda_j \geqslant \alpha_j \geqslant 0, j = 1, \ldots, k, \sum_{j=1}^{k} \lambda_j = 1 \right\} \text{ where } \sum_{j=1}^{k} \alpha_j \leqslant 1.$$

If the sum of lower bounds equals to one, say $\sum_{j=1}^{k} \alpha_j = 1$, we find only one valid extreme point $\boldsymbol{\lambda} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Otherwise, we make the change of variables $\mu_j = \lambda_j - \alpha_j \geqslant 0$, $j = 1, \ldots, k$, which consequently leads to a set M_{LB} in terms of μ_j, noting that the sum to unity constraint $\sum_{j=1}^{k} \lambda_j = 1$ is transformed into $\sum_{j=1}^{k} \mu_j = 1 - \sum_{j=1}^{k} \alpha_j > 0$:

$$\mathrm{M}_{LB} = \left\{ \boldsymbol{\mu} : \mu_j \geqslant 0, j = 1, \ldots, k, \sum_{j=1}^{k} \mu_j = 1 - \sum_{j=1}^{k} \alpha_j > 0 \right\}.$$

If we make a further change of variables $\gamma_j = \mu_j / (1 - \sum_{j=1}^{k} \alpha_j)$, $j = 1, \ldots, k$, the set M_{LB} is equivalently transformed to Γ_{LB} in terms of γ_j:

$$\Gamma_{LB} = \left\{ \boldsymbol{\gamma} : \gamma_j \geqslant 0, j = 1, \ldots, k, \sum_{j=1}^{k} \gamma_j = 1 \right\}.$$

Then we easily find the extreme points of Γ_{LB} as an identity matrix $\mathbf{I} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k)$ where $\mathbf{e}_j$ is a unit column vector whose jth element is one and zero elsewhere. To find the extreme points in terms of λ_j, which is our final goal, we multiply each $\mathbf{e}_j$ by a scalar $(1 - \sum_{j=1}^{k} \alpha_j)$ to maintain $\mu_j = (1 - \sum_{j=1}^{k} \alpha_j)\gamma_j$, thus yielding $(1 - \sum_{j=1}^{k} \alpha_j)\mathbf{e}_j$ and then we add a column vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)'$ to each $(1 - \sum_{j=1}^{k} \alpha_j)\mathbf{e}_j$ to maintain $\lambda_j = \mu_j + \alpha_j$. These operations finally produce the extreme points as follows: $\boldsymbol{\lambda}_1 = \left(\left(1 - \sum_{j=1, j\neq 1}^{k} \alpha_j\right), \alpha_2, \ldots, \alpha_k\right), \boldsymbol{\lambda}_2 = \left(\alpha_1, \left(1 - \sum_{j=1, j\neq 2}^{k} \alpha_j\right), \alpha_3, \ldots, \alpha_k\right), \ldots, \boldsymbol{\lambda}_{k-1} = \left(\alpha_1, \ldots, \alpha_{k-2}, \left(1 - \sum_{j=1, j\neq k-1}^{k} \alpha_j\right), \alpha_k\right), \boldsymbol{\lambda}_k = \left(\alpha_1, \ldots, \alpha_{k-1}, \left(1 - \sum_{j=1, j\neq k}^{k} \alpha_j\right)\right)$.

2.2. Rank ordered attributes (RO)

A set of rank ordered attributes, which is one of the most widely used types of incomplete attribute weights, is denoted by Λ_{RO}:

$$\Lambda_{RO} = \left\{ \boldsymbol{\lambda} : \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0, \sum_{j=1}^{k} \lambda_j = 1 \right\}.$$

To start with, we introduce new variables $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$ to denote $\mu_j = \lambda_j - \lambda_{j+1} \geqslant 0$, $j = 1, \ldots, k - 1$, $\mu_k = \lambda_k \geqslant 0$ and rearrange them to obtain $\lambda_k = \mu_k$, $\lambda_{k-1} = \mu_{k-1} + \mu_k$, $\lambda_{k-2} = \mu_{k-2} + \mu_{k-1} + \mu_k$, …, $\lambda_1 = \mu_1 + \mu_2 + \cdots + \mu_k$, noting that $\sum_{j=1}^{k} \lambda_j = \sum_{j=1}^{k} j \cdot \mu_j = 1$. These substitutions consequently lead to an equivalent set M_{RO} in terms of μ_j:

$$\mathrm{M}_{RO} = \left\{ \boldsymbol{\mu} : \mu_j \geqslant 0, j = 1, \ldots, k, \sum_{j=1}^{k} j \cdot \mu_j = 1 \right\}.$$

We now make a further change of variables $\gamma_j = j \cdot \mu_j$, $j = 1, \ldots, k$ to transform M_{RO} into Γ_{RO}, which enable us to obtain a set of extreme points with ease

[1] Some authors refer to it as ordered metric (Kmietowicz & Pearman, 1982; Pearman, 1993).