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Orderings of coherent systems with randomized dependent components

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ABSTRACT

Consider a general coherent system with independent or dependent components, and assume that the components are randomly chosen from two different stocks, with the components of the first stock having better reliability than the others. Then here we provide sufficient conditions on the component's lifetimes and on the random numbers of components chosen from the two stocks in order to improve the reliability of the whole system according to different stochastic orders. We also discuss several examples in which such conditions are satisfied and an application to the study of the optimal random allocation of components in series and parallel systems. As a novelty, our study includes the case of coherent systems with dependent components by using basic mathematical tools (and copula theory).

1. Introduction

Coherent systems are basic concepts in reliability theory (see, e.g., Barlow & Proschan (1981) and Kuo & Zhu (2012) for a detailed introduction to coherent systems, related properties and applications). Series systems, parallel systems and k-out-of-n systems are particular cases of coherent systems. In the theory of coherent systems, it is important to study the performance of a system composed by different kinds of units, and to define the optimal allocation of these units in the system. Some results on this topic are given in da Costa Bueno (2005), da Costa Bueno and do Carmo (2007), Li and Ding (2010), Brito, Zequeira, and Valdés (2011), Misra, Dhariyal, and Gupta (2009), Navarro and Rychlik (2010), Zhang (2010), Eryilmaz (2012), Kuo and Zhu (2012), Zhao, Chan, Li, and Ng (2013), Zhao, Chan, and Ng (2012), Belzunce, Martínez-Puertas, and Ruiz (2013), Levitin, Xing, and Dai (2014), and Hazra and Nanda (2014). In this context, a translation of the Parrondo's paradox was proposed in Di Crescenzo (2007). Parrondo's paradox shows that, in game theory, sometimes a random strategy might be a better option than any deterministic strategy. Analogously, Di Crescenzo (2007) proves that in a series system with independent non-identically distributed components, the random choice of these components is a better option than to use components with different distributions in the system. Indeed, Di Crescenzo (2007) proved that, in some situations, a random choice is the best option when we have to choose between two kinds of units having different behaviors. This result was extended to other system structures and to the case of dependent components in Navarro and Spizzichino (2010). More recently, the analysis of series and parallel systems formed by components having independent lifetimes and randomly chosen from two different stocks has been performed in Di Crescenzo and Pellerey (2011). In that paper, they assume that the components can be chosen from two stocks, where the items of a batch are better than those of the other one in the usual stochastic order. Under these assumptions, they obtain conditions on the random numbers of components chosen from the two stocks such that the reliability of system's lifetime improves. Similar results were obtained recently in Hazra and Nanda (2014) for other stochastic orders and for series and parallel systems with independent components.

In this paper we extend the results given in Di Crescenzo and Pellerey (2011) and Hazra and Nanda (2014) for series and parallel systems with independent components to general coherent systems with arbitrary structures and with possibly dependent components. We also show that these results can be used to study the optimal random allocation in a coherent system of components chosen from two different stocks.

The rest of the paper is organized as follows. In Section 2 we recall useful notions and definitions, such as the definitions of the stochastic orders used to compare the performances of systems, and the notion of copula, which is used to formally describe the dependence between the component lifetimes. We also

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describe the model of coherent systems with a random number of components chosen from different stocks. Section 3 contains the main results, which are centered on conditions to improve these systems in the stochastic, hazard rate, reversed hazard rate and likelihood ratio orders. In Section 4 we study ordering properties for systems in which the units are chosen randomly by means of Bernoulli trials. Several examples are provided in Section 5 to illustrate our theoretical results. In particular, certain general results are obtained for series systems when the dependence between the components is modeled by an Archimedean copula. An application to the study of the optimal random allocation of components in series and parallel systems is given in Section 6. Finally, some concluding remarks are presented in Section 7.

Hereafter some conventions and notations used throughout the paper are given. The notation $=_{st}$ stands for equality in distribution. We write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing". Also, we will denote by $D_i g(x_1,\dots,x_m)$ the partial derivative with respect to x_i of the function g, i.e.,

$$D_i g(x_1,\dots,x_m) = \frac{\partial}{\partial x_i} g(x_1,\dots,x_m)$$

whenever it exists. Moreover, we shall use the usual notation (a,b) for the open interval of real numbers $(a,b)=\{x\in\mathbb{R} : a<x<b\}$. Analogously, $[a,b]$ represents the closed interval.

2. Preliminaries

Firstly, we briefly recall the definitions of the stochastic orders that will be used throughout the paper to compare random lifetimes or the random numbers of components chosen from the two stocks. For further details, basic properties and applications of these orders, we refer the reader to Shaked and Shanthikumar (2007), or to Barlow and Proschan (1981) where, in particular, a list of applications in reliability theory is described.

Let X and Y be two absolutely continuous random variables having common support $[0,\infty)$, distribution functions F and G, and reliability (or survival) functions $\overline{F}=1-F$ and $\overline{G}=1-G$, respectively. Let f and g be their probability density functions, $\lambda_X = f/\overline{F}$ and $\lambda_Y = g/\overline{G}$ be their hazard functions and $\tilde{\lambda}_X = f/F$ and $\tilde{\lambda}_Y = g/G$ be their reversed hazard functions, respectively. We say that X is smaller than Y

- in the *usual stochastic order* (denoted by $X\leq_{st}Y$) if $\overline{F}(t)\leqslant\overline{G}(t)$ for all $t\in[0,\infty)$,
- in the *hazard rate order* (denoted by $X\leq_{hr}Y$) if $\lambda_X(t)\geqslant\lambda_Y(t)$ for all $t\in[0,\infty)$, i.e., if the ratio $\overline{F}(t)/\overline{G}(t)$ is decreasing in $[0,\infty)$,
- in the *reversed hazard rate order* (denoted by $X\leq_{rhr}Y$) if $\tilde{\lambda}_X(t)\leqslant\tilde{\lambda}_Y(t)$ for all $t\in[0,\infty)$, i.e., if the ratio $F(t)/G(t)$ is decreasing in $[0,\infty)$,
- in the *likelihood ratio order* (denoted by $X\leq_{lr}Y$) if the ratio $f(t)/g(t)$ is decreasing in $[0,\infty)$,
- in the *convex order* (denoted by $X\leq_{cx}Y$) if $E(\psi(X))\leqslant E(\psi(Y))$ for all convex functions ψ for which both expectations exist,
- in the *increasing convex order* (denoted by $X\leq_{icx}Y$) if $E(\psi(X))\leqslant E(\psi(Y))$ for all increasing convex functions ψ for which both expectations exist,
- in the *increasing concave order* (denoted by $X\leq_{icv}Y$) if $E(\psi(X))\leqslant E(\psi(Y))$ for all increasing concave functions ψ for which both expectations exist.

In particular here we just point out that

- $X\leq_{hr}Y$ if, and only if, $[X-t \mid X>t]\leq_{st}[Y-t\mid Y>t]$ for all $t\in[0,\infty)$,
- $X\leq_{rhr}Y$ if, and only if, $[t-X\mid X\leqslant t]\geq_{st}[t-Y\mid Y\leqslant t]$ for all $t\in[0,\infty)$,
- $X\leq_{lr}Y$ if, and only if, $[X\mid a\leqslant X\leqslant b]\leq_{st}[Y\mid a\leqslant X\leqslant b]$ for all $a\leqslant b$.

Hence, the hazard rate order and the reversed hazard rate order are often employed to compare residual lifetimes and inactivity times of systems, respectively, and the likelihood ratio order can be used to compare both residual lifetimes and inactivity times, while this is not the case for the weaker usual stochastic order. Moreover, the following relationships are well known:

$$\begin{array}{ccccc} X\leq_{lr}Y & \Rightarrow & X\leq_{hr}Y & & \\ \Downarrow & & \Downarrow & & \\ X\leq_{rhr}Y & \Rightarrow & X\leq_{st}Y & \Rightarrow & X\leq_{icx,icv}Y. \end{array}$$

Dealing with vectors of possibly dependent lifetimes, a common tool to describe the dependence is by means of its copula. Given a vector $\boldsymbol{X}=(X_1,\dots,X_m)$ of lifetimes, having joint distribution F and marginal distributions $F_1,\dots,F_m$, the function $C:[0,1]^m\to\mathbb{R}^+$ is said to be the *copula* of $\boldsymbol{X}$ if

$$F(x_1,\dots,x_m)=C(F_1(x_1),\dots,F_m(x_m)),\quad \text{for all } (x_1,\dots,x_m)\in\mathbb{R}^m.$$

If the marginal distributions F_i, for $i=1,\dots,m$, are continuous, then the copula C is unique and it is defined as

$$\begin{aligned} C(u_1,\dots,u_m) &= F\left(F_1^{-1}(u_1),\dots,F_m^{-1}(u_m)\right) \\ &= P(F_1(X_1)\leqslant u_1,\dots,F_m(X_m)\leqslant u_m) \end{aligned}$$

for $u_1,\dots,u_m\in(0,1)$. Copulas entirely describe the dependence between the components of a random vector; for example, concordance indexes like the Spearman's ρ or Kendall's τ of a vector $\boldsymbol{X}$ can be defined by means of its copula (see Nelsen (2006) for a monograph on copulas and their properties).

In reliability, often the survival copula S instead of the copula C is considered. Let $\boldsymbol{X}$ have joint reliability function $\overline{F}$ and marginal reliability functions $\overline{F}_1,\dots,\overline{F}_m$; then the function $S:[0,1]^m\to\mathbb{R}^+$ is said to be the *survival copula* of $\boldsymbol{X}$ if

$$\overline{F}(x_1,\dots,x_m)=S(\overline{F}_1(x_1),\dots,\overline{F}_m(x_m)),\quad \text{for all } (x_1,\dots,x_m)\in\mathbb{R}^m.$$

Among copulas (or survival copulas), particularly interesting is the class of Archimedean copulas. A copula is said to be *Archimedean* if it can be written as

$$C(u_1,\dots,u_m)=\overline{W}\left(\sum_{i=1}^m \overline{W}^{-1}(u_i)\right)\quad \text{for all } u_i\in[0,1] \tag{2.1}$$

for a suitable decreasing and m-monotone function $\overline{W}:[0,\infty)\to[0,1]$ such that $\overline{W}(0)=1$ and with inverse function $\overline{W}^{-1}$. The function $\overline{W}$ is usually called the *generator* of the Archimedean copula C. As pointed out in Nelsen (2006), many standard distributions (such as the ones in Gumbel, Frank, Clayton and Ali–Mikhail–Haq families) are special cases of this class. We also refer the reader to Müller and Scarsini (2005) or McNeil and Nešlehová (2009), and references therein, for details, properties and recent applications of Archimedean copulas. All the Archimedean copulas are *exchangeable*, that is,

$$C(u_1,\dots,u_m)=C(u_{\sigma(1)},\dots,u_{\sigma(m)})$$

for any permutation σ.

The model considered in this paper is described here. We consider a coherent system formed by m components, whose (possibly dependent) random lifetimes are denoted by $X_1,\dots,X_k,Y_{k+1},\dots,Y_m$. These lifetimes come from two distinct classes $\mathcal{C}_X=\{X_1,\dots,X_k\}$ and $\mathcal{C}_Y=\{Y_{k+1},\dots,Y_m\}$, having sizes k and $m-k$, respectively. Given two preassigned random lifetimes X and Y, we assume that the