European Journal of Operational Research 238 (2014) 166–174

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

The Quintile Share Ratio in location analysis

Tammy Drezner[a], Zvi Drezner[a,*], Beat Hulliger[b]

[a] Steven G. Mihaylo College of Business and Economics, California State University-Fullerton, Fullerton, CA 92834, United States
[b] School of Business, University of Applied Sciences Northwestern Switzerland, Switzerland

ARTICLE INFO

Article history:
Received 18 July 2013
Accepted 3 March 2014
Available online 13 March 2014

Keywords:
Quintile Share Ratio
Facility location
Equity
Global optimization

ABSTRACT

The inequality measure "Quintile Share Ratio" (QSR or sometimes S80/S20) is the primary income inequality measure in the European Union's set of indicators on social cohesion. An important reason for its adoption as a leading indicator is its simplicity. The Quintile Share Ratio is "The ratio of total income received by the 20% of the population with the highest income (top quintile) to that received by the 20% of the population with the lowest income (lowest quintile)". The QSR concept is used in this paper in the context of obnoxious facility location where the inequality is in distances to the obnoxious facility. The single facility location problem minimizing the QSR is investigated. The problem is investigated for continuous uniform demand in an area such as a disk, a rectangle, and a line; when demand is generated at a finite set of demand points; and when the facility can be located anywhere on a network. Optimal solution algorithms are devised for demand originating at a finite set of demand points and at nodes of the network. Computational experiments demonstrate the effectiveness of the algorithms.

1. Introduction

Typical location objectives are concerned with minimizing cost or maximizing efficiency of service. For example, the p-median objective minimizes the *average* service distance; the p-center objective minimizes the *maximum* service distance; the p-max-cover objective maximizes the number of customers served within a given distance. For a review of these problems see Current, Daskin, and Schilling (2002) and Daskin (1995). The objective of obnoxious facilities location models (Church & Garfinkel, 1978; Hansen, Peeters, & Thisse, 1981) is to locate a facility far from demand points. Facilities such as airports, prisons, nuclear power plants, dump sites, polluting plants and others are "obnoxious" and thus should be located away from population centers but should be close enough to provide the service or accommodate workers and visitors.

Many location models employ equity objectives (Berman, 1990; Berman & Kaplan, 1990; Hay, 1995; Lopez-De-Los-Mozos, Mesa, & Puerto, 2008; Mandell, 1991; Ogryczak, 2009; Ogryczak & Zawadzki, 2002). Equitable distance to service providers, equitable burden or nuisance imposed, etc. In the context of facility location the objective is to equalize as much as possible the distance between demand points and a facility that serves them. Maimon (1986) considered the minimization of the variance of the distances in a network environment. Drezner, Thisse, and Wesolowsky (1986) analyzed the minimization of the range of distances that customers travel in the plane. This is equivalent to locating a line that minimizes the maximum vertical distance to demand points (Schöbel, 1999). Drezner and Drezner (2007) analyzed minimizing both objectives – the variance of the distances and the range of distances in the plane. A commonly used measure of income inequity is the Gini index (Gini, 1921) of the Lorenz curve (Lorenz, 1905). In the location literature minimizing the Gini Coefficient is investigated in Maimon (1988), Drezner (2004) and Drezner, Drezner, and Guyse (2009). Eiselt and Laporte (1995) list 19 equity measures used in location models. Note that all these objectives attain their minimum value when all distances are equal. For a review of equity models in location see Drezner and Drezner (2007), Eiselt and Laporte (1995), Erkut (1993), Marsh and Schilling (1994) and Mulligan (1991).

Models which assign equitable loads to facilities were investigated for discrete planar problems (Drezner & Drezner, 2006), continuous planar demand (Baron, Berman, Krass, & Wang, 2007; Suzuki & Drezner, 2009), and the network environment (Berman, Drezner, Tamir, & Wesolowsky, 2009).

In the present paper we investigate the equity objective "Quintile Share Ratio" (QSR or sometimes S80/S20) which is the primary income inequality measure in the European Union's set of indicators on social cohesion (European Commission, 2003). An important reason for its adoption as a leading indicator is its simplicity. The Quintile Share Ratio is "The ratio of total income

* Corresponding author. Tel.: +1 (714) 278 2712.
E-mail addresses: tdrezner@fullerton.edu (T. Drezner), zdrezner@fullerton.edu (Z. Drezner), beat.hulliger@fhnw.ch (B. Hulliger).

http://dx.doi.org/10.1016/j.ejor.2014.03.001
0377-2217/

received by the 20% of the population with the highest income (top quintile) to that received by the 20% of the population with the lowest income (lowest quintile)" (Eurostat, 2012). Thus the definition is clear, short and readily understandable unlike the definition of other inequality measures like the Gini-Index (Drezner et al., 2009; Gini, 1921). In 2010 the QSR ranged from 3.4 for Norway and Slovenia to 7.3 for Lithuania (Eurostat, 2012).

Formally QSR is defined as follows: let $Y_1,\ldots,Y_n$ be independent identically distributed observations of a univariate variable on a sample of size n. In this article we assume that the sampling process is ignorable but in the context of the application of QSR as an inequality measure the sample usually is a complex survey sample and therefore special care for weighting and variance estimation must be observed (Langel & Tillé, 2011; Osier, 2009). Here we also assume that the Y_i are positive. Denote by $Y_{(1)} \leqslant Y_{(2)} \leqslant,\ldots,Y_{(n)}$ the order statistics. For any pair of probabilities p with $0 \leqslant p < 0.5$ and r with $0 \leqslant r < 0.5$ the quantile share ratio for p and r is defined as

$$Q(p,r) = \frac{\frac{1}{\lceil nr\rceil}\sum_{i=n+1-\lceil nr\rceil}^{n} Y_{(i)}}{\frac{1}{\lceil np\rceil}\sum_{i=1}^{\lceil np\rceil} Y_{(i)}} \qquad (1)$$

where $\lceil np\rceil$ is the smallest integer majorising np and similar for $\lceil nr\rceil$. The probabilities p and r are named for the proportion of poor and rich, respectively. The Quintile Share Ratio is $Q(p,r)$ for $p=r=0.2$. Obviously, the division by $\lceil np\rceil$ and $\lceil nr\rceil$ cancels out if $p=r$. Thus quantiles other than 20% and also differing quantiles for the numerator and denominator may be considered. However, for the purpose of this article we mainly use $p=r=0.2$. In addition, to avoid undue influence of outliers the top quintile share may be trimmed above. No trimming is considered in this paper but the suggested methods can be extended to trimmed values. Outlier-robustness of quantile shares has been considered in Cowell and Victoria-Feser (1996) and of the Quintile Share Ratio in Hulliger and Schoch (2009).

QSR is looking at the extremes of the distribution and is not sensitive to the middle part, as for example is the Gini-Index. The dependence on the involved quantiles, which must be estimated in a first step, and the ratio form make the QSR a non-linear estimator and its statistical properties must be investigated carefully (Langel & Tillé, 2011). However, the QSR and other inequality measures have a monotonic relationship for important income distribution functions (Graf, Nedyalkova, Münnich, Seger, & Zins, 2011) and are capable of transmitting similar messages. In this paper we treat service distance the same way income is treated in wealth distribution.

2. Formulation

In the location context the formulation is based on n demand points located at given locations in a plane, or on nodes of a network. For a given facility location X the distances vector between the demand points and the facility is $\{d_i(X)\}$ for $i=1,\ldots,n$. The sorted vector of distances is defined as $d_{(1)}(X) \leqslant \cdots \leqslant d_{(n)}(X)$. For simplicity of presentation we develop the formulas for equal weights for all demand points. The general weight case requires a small modification of the un-weighted case and is described in Section 4.3.

In the context of obnoxious facility location a farther location is better. Therefore, demand points at large distances are "rich" and demand points close to the facility are disadvantaged and thus can be considered "poor". Our objective is equity of nuisance to customers. Equity is usually achieved when distances are very large (see, for example the discussion in Erkut (1993) and Drezner (2004)). At far away points all the distances are about the same and the limit of the QSR ratio, as the location of the facility moves to "infinity" is one. We should therefore restrict the location of the facility to a finite set such as the convex hull of demand points or a square that encloses all demand points. The average of the top r% distances is $\frac{1}{[nr]}\sum_{i=n+1-[nr]}^{n} d_{(i)}(X)$ where $[x]$ is the rounded integer of x. One can use the ceiling of x for the same purpose but if n is large it does not matter much. In all our experiments we selected n such that both nr and np are integer. A similar expression is formulated for the bottom p% of distances. The ratio to be minimized is:

$$Q(p,r,X) = \frac{\frac{1}{[nr]}\sum_{i=n+1-[nr]}^{n} d_{(i)}(X)}{\frac{1}{[np]}\sum_{i=1}^{[np]} d_{(i)}(X)} \qquad (2)$$

$QSR(X)$ is defined as $Q(0.2,0.2,X)$.

3. Continuous demand

We investigate the location of an obnoxious facility when demand is uniformly distributed in a finite area of the plane. We analyze location in a disk, a rectangle, and a segment. The results are sensitive to the uniformity of demand. If the distribution of the demand is not uniform, different conclusions may be reached.

3.1. Location in a disk

Consider an infinite number of demand points uniformly distributed in a disk (a circle and its interior). First suppose that a facility is located at the center of the circle. Consider a ring based on concentric circles of inner radius R_1 and outer radius R_2. By integration, the average distance from the center of the concentric circles to all demand points in the ring is

$$\frac{2(R_1^2+R_1R_2+R_2^2)}{3(R_1+R_2)} \qquad (3)$$

The circle containing the closest p% of distances has a a radius $\sqrt{p}$ and thus $R_1=0,\ R_2=\sqrt{p}$ leading to an average distance by (3) of $\frac{2}{3}\sqrt{p}$. The farthest r% demand points are in a ring with $R_1=\sqrt{1-r},\ R_2=1$ leading by (3) to an average distance of

$$\frac{2(1+\sqrt{1-r}+1-r)}{3(1+\sqrt{1-r})} = \frac{2(1-(1-r)\sqrt{1-r})}{3r} \qquad (4)$$

and the ratio is

$$\frac{1-(1-r)\sqrt{1-r}}{r\sqrt{p}} \qquad (5)$$

When the location X of the facility diverges to infinity, $\lim_{X\to\infty} Q(p,r,X)=1$.

In the rest of this section we analyze $p=r=20\%$. At the center of the circle, by (5), $QSR(0)=5\sqrt{5}-8\approx 3.18$. We calculated and plotted the graph of $QSR(X)$ for X on the positive x-axis, and a demand circle of radius 1 centered at the origin. The graph is depicted in Fig. 1. It is clear from the graph that the minimum QSR inside the circle is achieved at the center of the circle. It increases up to about 71% of the radius reaching a high of 4.78, dropping to 3.94 at the periphery of the circle. A ratio below 3.18 is obtained outside the circle for a distance greater than 1.21. If the location of the facility is allowed anywhere in the square bounding the circle, the optimal location is at the vertex of the square because its distance from the center of the disk is $\sqrt{2}>1.21$.

It is interesting that QSR has a very similar shape as the Gini coefficient reported in Drezner et al. (2009). They found that when the facility is located at the center of a disk, the Gini coefficient is equal to 0.2. It increases up to about 65% of the radius of the disk and then decreases. On the periphery of the disk, the Gini coefficient is 0.2376.