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Discrete Optimization

On connected dominating sets of restricted diameter

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ABSTRACT

A connected dominating set (CDS) is commonly used to model a virtual backbone of a wireless network. To bound the distance that information must travel through the network, we explicitly restrict the diameter of a CDS to be no more than s leading to the concept of a dominating s-club. We prove that for any fixed positive integer s it is NP-complete to determine if a graph has a dominating s-club, even when the graph has diameter $s+1$. As a special case it is NP-complete to determine if a graph of diameter two has a dominating clique. We then propose a compact integer programming formulation for the related minimization problem, enhance the approach with variable fixing rules and valid inequalities, and present computational results.

1. Introduction

It is common to represent a wireless communication network as a graph, where nodes in the wireless network correspond to vertices in the graph and edges denote the ability to communicate directly. Any given pair of nodes may not be close enough to communicate directly, but a message can be transmitted through intermediate nodes. In the wireless network, these intermediate nodes form a virtual backbone structure, which is a connected dominating set (CDS) in the graph setting. It may be important to transmit these messages quickly. This requirement of low-latency can be enforced in the graph by requiring that the vertices from the CDS induce a subgraph of small diameter. This leads to the notion of a dominating s-club—a dominating set whose induced subgraph has diameter at most s. It can be seen that dominating s-club generalizes dominating clique (let $s=1$), and if we allow s to be instance-dependent it generalizes CDS (let $s=n-1$ for an n-vertex graph).

A dominating clique (i.e., a dominating 1-club) introduced by Brandstädt and Kratsch (1985) (see also Brandstädt & Kratsch, 1987; Cozzens & Kelleher, 1990) can provide a virtual backbone structure for a wireless communication network that has the shortest communication time. Since the members of the dominating clique can communicate directly, a message departing from any source can reach any destination in at most 3 hops. On the other hand, ensuring the existence of a dominating clique in a wireless network may be too costly. Moreover, this may result in excessive interference.

A dominating s-club provides a compromise, possibly offering advantages over the commonly used CDS in terms of speed of communication, energy consumption, and reliability (Kim, Wu, Li, Zou, & Du, 2009). Small diameter dominating sets facilitate quick communication between each pair of vertices, as any two vertices are at most $s+2$ hops apart: 1 hop to reach the backbone, s hops within the backbone, and 1 hop to the destination. All else being equal, smaller dominating sets contribute to improved energy consumption, as the number of transmissions required to pass a message from one vertex to another is small. Also recognize that long transmission paths can increase the chance of message transmission failure (Mohammed, Gewali, & Muthukumar, 2005), meaning that a smaller-diameter backbone may be more reliable. In contrast to a dominating s-club, a CDS D ensures little in terms of the diameter (the worst case diameter is $|D|-1$), possibly resulting in excessively long transmission paths between vertices.

The minimum dominating s-club problem, which is the focus of the present paper, is defined as follows. Given a graph $G=(V,E)$ and a positive integer constant s, find a smallest dominating set $D \subseteq V$ such that $diam(G[D]) \leqslant s$, or decide that none exist. When a dominating s-club exists in G, we call the size of a minimum dominating s-club the *dominating s-club number* of G and denote it by $\gamma^s_{club}(G)$.

1.1. Notation and related work

We consider a simple undirected graph $G=(V,E)$ with set V of n vertices and set $E \subset V \times V$ of edges. We denote the (open) neighborhood of a vertex $i \in V$ by $N(i)=\{j \in V : \{i,j\} \in E\}$, and the

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closed neighborhood of $i \in V$ by $N[i] = N(i) \cup \{i\}$. A set $D \subseteq V$ is called a dominating set if each vertex in $V \setminus D$ has a neighbor in D, and a total dominating set if each vertex in V (including the vertices from D) has a neighbor in D. Let $G[S]$ be the graph induced by $S \subseteq V$. A dominating set that induces a connected graph is called a connected dominating set. Let the distance $d_G(i,j)$ be the length of a shortest path between vertices $i,j \in V$ in a graph G, and let $diam(G) = \max\{d_G(i,j) : i,j \in V\}$ be the diameter of G. We adopt the convention that $diam(G) = \infty$ for a disconnected graph G. A clique $C \subseteq V$ is a subset of pairwise adjacent vertices, i.e., $diam(G[C]) = 1$. Mokken and Cliques (1979) introduced a clique relaxation model called an *s-club*, which is a subset $S \subseteq V$ of vertices such that $diam(G[S]) \leqslant s$. An s-club that forms a dominating set is called a dominating s-club.

The minimum dominating clique problem has been shown to be polynomial-time solvable in strongly chordal graphs (Kratsch, 1990), undirected path graphs (Kratsch, 1990), and circle graphs (Keil, 1993). In general, however, the minimum dominating clique problem is NP-hard (Kratsch & Liedloff, 2007). In fact, it is NP-complete to determine the existence of a dominating clique in a graph (Brandstädt & Kratsch, 1985, 1987). An exact, exponential-time algorithm has been developed for arbitrary graphs (Kratsch & Liedloff, 2007).

Since we want to restrict the diameter, the notion of s-club is important. Several papers in literature consider a different problem: the maximum s-club problem. Even though this problem is different from ours, it provides useful insights for complexity and IP formulations. The maximum s-club problem asks for a maximum size subset of vertices whose induced subgraph has diameter at most s. The decision version of the maximum s-club problem has been shown to be NP-complete (Bourjolly, Laporte, & Pesant, 2002), even when restricted to graphs of diameter $s+1$ (Balasundaram, Butenko, & Trukhanov, 2005). Some exact approaches for the maximum s-club problem are based on integer programming formulations with $O(n^{s+1})$ entities (Balasundaram et al., 2005; Bourjolly et al., 2002). More recently, Veremyev and Boginski (2012) proposed a compact formulation having $O(sn^2)$ entities. This compact formulation is key to the development of the formulation proposed in this paper, which also has $O(sn^2)$ entities. The reader is referred to Pattillo, Youssef, and Butenko (2013) for more information on s-clubs and other clique relaxation models.

The NP-hard minimum connected dominating set (MCDS) problem and the related maximum leaf spanning tree problem have received significant attention in the literature. A wide variety of approaches have been considered, including exact approaches (Fan & Watson, 2012; Fernau et al., 2011; Fomin, Grandoni, & Kratsch, 2008; Fujie, 2003; Fujie, 2004; Lucena, Maculan, & Simonetti, 2010; Simonetti, Salles da Cunha, & Lucena, 2011), approximation algorithms (Guha & Khuller, 1998; Lu & Ravi, 1998), and polynomial-time approximation schemes for unit disk graphs (Cheng, Huang, Li, Wu, & Du, 2003; Hunt et al., 1998) and unit ball graphs (Zhang, Gao, Wu, & Du, 2008).

Considerably less work has been done for restricted-diameter dominating sets. Researchers have proposed heuristics (Mohammed et al., 2005) and approximation algorithms (Kim et al., 2009) with constant ratios for both CDS size and diameter (for disk graphs). Schaudt (2013) has shown that it is NP-complete to determine if a graph of diameter $s+2$ has a dominating s-club. To the best of our knowledge, there are no exact approaches for the minimum dominating s-club problem in previous literature (for general s).

1.2. Our contributions

In Section 2, we prove that it is NP-complete to determine if a graph of diameter two has a dominating clique. We extend this for any fixed positive integer s, showing that it is NP-complete to determine if a graph of diameter $s+1$ admits a dominating s-club. We consider the minimization problem in Section 3, showing that it is NP-hard to approximate within a logarithmic factor, even when a dominating s-club is known to exist. We also relate the dominating s-club number to other domination-related graph invariants, and demonstrate that restricting the diameter by just one unit can be very costly. In Section 4, we present a compact integer programming formulation for the minimum dominating s-club problem and develop associated valid inequalities and variable fixing rules. Section 5 discusses the potential applicability of the formulation to solving the classical MCDS problem. Section 6 reports the results of numerical experiments with the proposed formulation for the minimum dominating s-club problem on a set of random unit disk graphs and MCDS instances from literature. Finally, Section 7 concludes the paper.

2. Existence of a dominating *s*-club

In this section we consider the questions: (1) which graphs have a dominating s-club? and (2) what is the computational complexity of determining if a graph admits a dominating *s-club?*

It is clear that not every graph has a dominating s-club, as evidenced by the class of disconnected graphs. However, even when the graph is connected, a dominating s-club may not exist; the cycle on six vertices has no dominating 2-club. Some have attempted to find sufficient conditions for a dominating s-club to exist. Cozzens and Kelleher (1990) show that a dominating clique exists if the graph has no induced C_5 and no induced P_5. (P_t and C_t are the path and cycle on t vertices, respectively.) See also (Bacsó & Tuza, 1990). This has been extended for other values of s (Bacsó, 2009; Bacsó & Tuza, 1997; Tuza, 2008). Namely, for $s=1$ or $s=2$, a dominating s-club will exist if the graph has no induced C_{s+4} and no induced P_{s+4}; for $s \geqslant 3$, a dominating s-club will exist if there is no induced P_{s+4} (Tuza, 2008).

In the following proposition, we note some simple cases where a dominating s-club will or will not exist.

Proposition 1. *Consider a graph $G = (V,E)$.*

1. *For any $s \geqslant diam(G)$, there exists a dominating s-club in G.*
2. *If $diam(G) \geq 4$ then for any $s < diam(G) - 2$, there does not exist a dominating s-club in G.*

Proof. It is clear that V is a dominating s-club in the first case. To prove the second case, consider vertices $v, v' \in V$ such that $d_G(v, v') = diam(G)$. We claim that a dominating s-club must include $u \in N(v)$ and $u' \in N(v')$. (Otherwise, v (or v') must be dominated by itself, in which case v (or v') is isolated and the dominating set is not connected.) Thus, for any dominating set $S \subseteq V$, $d_{G[S]}(u,u') \geqslant d_G(u,u') \geqslant diam(G) - 2 > s$, implying that u and u' cannot belong to the same s-club. $\square$

This leaves two values of s where it is not yet clear if a dominating s-club exists or not: $s = diam(G) - 1$ and $s = diam(G) - 2$. It turns out that these cases are NP-complete. The case $s = diam(G) - 2 = 1$ (i.e., DOMINATING CLIQUE in diameter 3 graphs) has been shown by Brandstädt and Kratsch (1987). We and Schaudt (2013) independently show that the problem is NP-complete for *each* fixed positive integer s. We note however that the reduction used by Schaudt (2013) has $s = diam(G) - 2$. Our reduction allows us to make a different statement—that the NP-completeness holds when $s = diam(G) - 1$. First, we show that determining if a diameter two graph has a dominating clique is NP-complete. Then, using this fact, we describe NP-completeness reductions for even $s \geqslant 2$ and odd $s \geqslant 3$ (both are provided in the appendix).