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Decision Support

On a class of solidarity values

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ABSTRACT

We suggest a new one-parameter family of solidarity values for TU-games. The members of this class are distinguished by the type of player whose removal from a game does not affect the remaining players' payoffs. While the Shapley value and the equal division value are the boundary members of this family, the solidarity value is its center. With exception of the Shapley value, all members of this family are asymptotically equivalent to the equal division value in the sense of Radzik (2013).

1. Introduction

A cooperative game with transferable utility (TU-game) consists of a non-empty and finite set of players, N, and a coalition function, $v : 2^N \to \mathbb{R},\ v(\emptyset) = 0$, which for each coalition $S \subseteq N$ describes the worth $v(S)$ that can be generated by its members. Assuming that the grand coalition eventually is formed, the question arises how to distribute the grand coalition's worth, $v(N)$. The most prominent single-point solution concept that answers this question probably is the Shapley value (Shapley, 1953).

While almost all modern societies reveal some degree of solidarity, the Shapley value does not allow for solidarity among the players. Unproductive players (null players) are not only assigned zero payoffs, but may even leave the game without affecting the other players' payoffs. Moreover, a player's payoff depends only on his *own* marginal contributions. In contrast, the equal division value distributes the worth generated by the grand coalition equally among the players. This can be interpreted as an extreme kind of solidarity. Clearly, the payoffs according to the equal division value are almost insensitive to a player's own marginal contributions.

Several attempts have been made to provide solution concepts that live in between these two extremes. Sprumont (1990) suggests an example of a population monotonic allocation scheme, later on characterized by Nowak and Radzik (1994) as the solidarity value. Nowak and Radzik (1996) study the convex mixtures of the Shapley value and their solidarity value. The class of egalitarian Shapley values, i.e., convex mixtures of the Shapley value and the equal division value is suggested by Joosten (1996). Joosten (1996) and Driessen and Radzik (2002) come up with the discounted Shapley values. The whole class of efficient, linear, and symmetric values is described by Chameni Nembua (2012), for example.

Kamijo and Kongo (2010, 2012) show that the difference between the Shapley value, the equal division value, and the solidarity value can be pinpointed to just one axiom. The aforementioned values differ in an axiom that specifies the type of player that can be removed from the player set without affecting the remaining players' payoffs. For the Shapley value, null players can be excluded. Proportional players can be eliminated for the equal division value, where a proportional player is a player whose entrance to a coalition does not change the per capita worth. Closely related is the notion of a quasi-proportional player, which can be removed for the solidarity value.

In this paper, we suggest a family of player types—the ξ-players—that contains the null players, the proportional players, and quasi-proportional players. A ξ-player is a player whose marginal contribution to a coalition is ξ times the per capita worth of the coalition he enters. The value of ξ may depend on the size of the coalition entered, i.e., ξ actually is sequence of real numbers. For example, a null player is a ξ-player if ξ is constantly zero and a proportional player is a ξ-player if ξ is constantly one. The notion of a ξ-player gives rise to the corresponding ξ-player out axiom. As our first result, we determine those sequences ξ for which there

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exists a value that satisfies efficiency and the ξ-player out axiom. Further, we show that for these sequences there exits a unique value that satisfies efficiency, linearity, symmetry, and the ξ-player out axiom.

The family of these values, parametrized by ξ, contains values that are not economically sound. In particular, these values may fail desirability (Maschler & Peleg, 1966) or positivity (Kalai & Samet, 1987). Desirability ensures that a player who is more productive than another one does not end up with a lower payoff. Positivity requires that all players obtain non-negative payoffs in monotonic games, i.e., in games, where all marginal contributions are non-negative. Radzik and Driessen (2013) give necessary and sufficient conditions for a general value to satisfy these two properties. As our second result, we employ these conditions to identify the sequences ξ for which the values in our family meet desirability and positivity. The values in this subfamily are called the "generalized solidarity values". Except the Shapley value and the equal division value, all other egalitarian Shapley values are not members of this subfamily. The solidarity value is the center of the family of generalized solidarity values, which indicates that the name of this class is appropriate.

Radzik (2013) establishes that the solidarity value and the equal division value are asymptotically equivalent in two respects. Let us explain the weaker one. Consider a sequence of games such that the set of players strictly increases, but the worth generated by any coalition of the these games is bounded. Fix a player in some of these games. Then, the difference of this player's payoff for the solidarity value and for the equal division values converges to zero as the size of the player set goes to infinity. Our third result says that the generalized solidarity values with exception of the Shapley value, also are asymptotically equivalent to the equal division value in the sense of Radzik (2013).

The plan of this paper is as follows: In Section 2, basic definitions and notation are given. In Section 3, we analyze the player out properties and introduce the generalized solidarity values. In Section 4, we establish the asymptotic equivalence of the generalized solidarity values and the equal division value. Some remarks conclude the paper. Appendix A contains all the proofs.

2. Basic definitions and notation

Fix a countably infinite set $\mathfrak{U}$, the universe of players, and let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathfrak{U}$. For $R, S, T, N \in \mathcal{N}$, r, s, t, and n denote their cardinalities, respectively. A **(TU)-game** is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v \in \mathbb{V}(N) := \{f : 2^N \to \mathbb{R} | f(\emptyset) = 0\}$, where 2^N denotes the power set of N. Sometimes, for notational parsimony, we will write v^N instead of $v \in \mathbb{V}(N)$. Also for notational convenience, we will write the singleton $\{i\}$ as i.

Subsets of N are called **coalitions**, and $v^N(S)$ is called the worth of coalition S. For $v, w \in \mathbb{V}(N)$, $\rho \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}(N)$ and $\rho \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\rho \cdot v)(S) = \rho \cdot v(S)$ for all $S \subseteq N$. For $S \subseteq N$ and $v \in \mathbb{V}(N)$, $v|_S \in \mathbb{V}(S)$ denotes the restriction of v to 2^S. For $T \subseteq N$, $T \neq \emptyset$, the game u_T^N, $u_T^N(S) = 1$ if $T \subseteq S$ and $u_T^N(S) = 0$ otherwise, is called a **unanimity game**; the game e_T^N, $e_T^N(S) = 1$ if $T = S$ and $e_T^N(S) = 0$ otherwise, is called a **standard game**; the game $\mathbf{0}^N$, $\mathbf{0}^N(S) = 0$ for all $S \subseteq N$, is called a **null game**. Any $v \in \mathbb{V}(N)$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T^N, \quad \lambda_T(v) := \sum_{S \subseteq T : S \neq \emptyset} (-1)^{t-s} \cdot v(S). \tag{1}$$

Player $i \in N$ is called a **null player** in $v \in \mathbb{V}(N)$ if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$; players $i, j \in N$ are called **symmetric** in $v \in \mathbb{V}(N)$ if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

A **value** on $\mathcal{N}$ is an operator φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$. The **equal division value** is given by

$$\mathrm{ED}_i(v) = \frac{v(N)}{n}, \tag{2}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$. The **Shapley value** (Shapley, 1953) is given by

$$\mathrm{Sh}_i(v) = \sum_{S \subseteq N \setminus i} p_{n,s} \cdot (v(S \cup i) - v(S)) \tag{3}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$, where

$$p_{n,s} := \frac{1}{n} \cdot \binom{n-1}{s}^{-1}. \tag{4}$$

The **solidarity value** (Nowak & Radzik, 1994) is given by

$$\mathrm{So}_i(v) = \sum_{S \subseteq N : i \in S} \frac{p_{n,s-1}}{s} \cdot \sum_{j \in S} (v(S) - v(S \setminus j)) \tag{5}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$.

Below, we list the axioms that are referred to later on.

Efficiency, E. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Linearity, L. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$ and $\rho \in \mathbb{R}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $\varphi(\rho \cdot v) = \rho \cdot \varphi(v)$.

Null player, N. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$ and all $i \in N$ such that i is a null player in v, $\varphi_i(v) = 0$.

Symmetry, S. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$ and all $i, j \in N$ such that i and j are symmetric in v, $\varphi_i(v) = \varphi_j(v)$.

3. The family of generalized solidarity values

There is a multitude of values for TU-games on the market. A major purpose of axiomatic characterizations of values is to facilitate the decision which value to apply in a specific situation. Hence, it is of particular interest to pinpoint the difference between values to a single axiom in their characterizations.

Recently, Kamijo and Kongo (2010, 2012) provide characterizations of the Shapley value, the equal division value, and the solidarity value that differ in one axiom only. All characterizations employ the standards axioms efficiency, linearity, and symmetry.[1] The fourth axiom in these characterizations are properties that specify which type of player can leave a game without affecting the remaining players' payoffs. For the Shapley value, this is the null player.

Null player out (Derks & Haller, 1999), **NPO.** For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $i \in N$, and $j \in N \setminus i$ such that i is a null player in v, $\varphi_j(v|_{N \setminus i}) = \varphi_j(v)$.

For the equal division value and the solidarity value, they introduce the notion of a proportional player and a quasi-proportional player, respectively. Player $i \in N$ is called a **proportional player** in $v \in \mathbb{V}(N)$ if $v(i) = 0$ and

$$\frac{v(S \cup i)}{s+1} = \frac{v(S)}{s} \quad \text{for all } S \subseteq N \setminus i, \quad S \neq \emptyset;$$

player $i \in N$ is called a **quasi-proportional player** in $v \in \mathbb{V}(N)$ if

$$\frac{v(S \cup i)}{s+2} = \frac{v(S)}{s+1} \quad \text{for all } S \subseteq N \setminus i.$$

Proportional player out, PPO. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $i \in N$, and $j \in N \setminus i$ such that i is a proportional player in v, $\varphi_j(v|_{N \setminus i}) = \varphi_j(v)$.

[1] Actually, they employ the balanced cycle contributions axiom instead of linearity and symmetry. Since the mentioned values satisfy linearity and symmetry, and since by Kamijo and Kongo (2012, Theorem 1), all symmetric and linear values already satisfy balanced cycle contributions, the claim follows.