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Discrete Optimization

Locating a single facility and a high-speed line

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ABSTRACT

In this paper we study a facility location problem in the plane in which a single point (facility) and a rapid transit line (highway) are simultaneously located in order to minimize the total travel time from the clients to the facility, using the L_1 or Manhattan metric. The rapid transit line is given by a segment with any length and orientation, and is an alternative transportation line that can be used by the clients to reduce their travel time to the facility. We study the variant of the problem in which clients can enter and exit the highway at any point. We provide an $O(n^3)$-time algorithm that solves this variant, where n is the number of clients. We also present a detailed characterization of the solutions, which depends on the speed given along the highway.

1. Introduction

Suppose we are given a set of clients represented as a set of points in the plane, and a service facility represented as a point to which all clients have to move. Every client can reach the facility directly, or use an alternative rapid transit line called highway in order to reduce the travel time. The highway is a straight line segment of arbitrary orientation and length. If a client moves directly to the facility, it moves at unit speed and the distance traveled is the Manhattan or L_1 distance to the facility. In the case where a client uses the highway, it travels the L_1 distance at unit speed to one point of the highway, traverses with a speed $v > 1$ the Euclidean distance to another highway point, and finally travels the L_1 distance from that point to the facility at unit speed. All clients traverse the highway at the same speed. The highway is used by a client point whenever it saves time to reach the facility. Given the set of points representing the clients, the facility location problem consists in determining at the same time the facility point and the highway in order to minimize the total weighted travel time from the clients to the facility. The weighted travel time of a client is its travel time multiplied by a weight representing the intensity of its demand.

Geometric problems related to transportation networks have been recently considered, and simplified mathematical models have been widely studied in order to investigate basic geometric properties of urban transportation systems. Abellanas et al. (2003) introduced the time metric model: Given an underlying metric, the user can travel at speed $v(h)$ when moving along a highway h or unit speed elsewhere. The particular case in which the underlying metric is the L_1 metric and all highways are axis-parallel segments of the same speed, is called the *city metric* (Aichholzer, Aurenhammer, & Palop, 2002; Görke, Shin, & Wolff, 2008). The optimal positioning of transportation systems that minimize the maximum travel time among a set of points has been investigated in detail in recent papers (Ahn et al., 2007; Aloupis et al., 2010; Cardinal, Collette, Hurtado, Langerman, & Palop, 2008; Cardinal, Labbé, Langerman, & Palop, 2009). Other similar and more general models are studied by Bae, Korman, and Tokuyama (2009) and Korman and Tokuyama (2008).

A similar problem of simultaneously locating a service facility point and a highway of fixed length was recently studied by Espejo and Rodríguez-Chía (2011, 2012) and Díaz-Báñez, Korman, Pérez-Lantero, and Ventura (2013). The authors considered locating a *turnpike* (Bae et al., 2009), that is, a highway in which clients can enter and exit the highway only at the endpoints. A first solution running in $O(n^3 \log n)$ time, where n denotes the number of clients, was introduced by Espejo and Rodríguez-Chía (2011, 2012), and an $O(n^3)$-time improved one was given by Díaz-Báñez, Korman, Pérez-Lantero, and Ventura (2012b). The problem aims to minimize the total weighted travel time from the demand points to the facility

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service. Díaz-Báñez, Korman, Pérez-Lantero, and Ventura (2012a) continued the study of this variant by considering the min–max optimization criterion. They minimize the maximum time distance from the clients to the facility point.

1.1. Notation

The following notation is introduced in order to formulate the problem. Let S be the set of n demand points, f be the service facility point, h be the highway, and $v > 1$ be the speed in which demand points move along h. Given a demand point p, $w_p > 0$ denotes the weight of p. The travel time between a demand point p and the service facility f is denoted by

$$d_h(p,f) := \min\left\{\|p-f\|_1, \min_{q_1,q_2\in h}\left\{\|p-q_1\|_1 + \frac{\|q_1-q_2\|_2}{v} + \|q_2-f\|_1\right\}\right\} \quad (1)$$

The problem can be formulated as follows:

1.2. The Freeway and Facility Location problem (FFL-problem)

Given a set S of n demand points, the weight $w_p > 0$ of each point p of S, and fixed speed $v > 1$, locate the facility point f and the highway h that minimizes the function

$$\Phi(f,h) := \sum_{p\in S} w_p \cdot d_h(p,f). \quad (2)$$

We consider the case in which a highway is a straight object (that is, a segment or a line). Note that we have complete liberty on where to locate f, whereas the location of S (and the value of v) is fixed by the problem instance.

1.3. Motivation

This metric tries to model the time needed for a person to travel between any two points in a modern city. In most cases, a person walks (or drives) on streets, thus a path usually consists of orthogonal (i.e., north–south or east–west) roads. Thus, we use L_1 as the underlying metric. However, there are a few exceptional non-orthogonal roads that one can use as a short-cut (i.e., Broadway Avenue in Manhattan). These roads make non-orthogonal paths feasible (thus, we allow L_2 metric within the highway). Moreover, people on these roads are normally given priority, hence we can even travel faster while on these roads (i.e., $v > 1$). This type of non-orthogonal road is often called *freeway* (Bae et al., 2009) or simply *highway* (Korman & Tokuyama, 2008) in the literature.

Shortest path computation within a city with several highways (and even possibly obstacles) is well understood (Bae et al., 2009). However, the constructive variations of this problem (i.e., deciding where to locate a highway) have been barely studied. As mentioned before, there has been an interest in simultaneously locating a facility and a highway (Díaz-Báñez et al., 2012a, 2012b; Espejo & Rodríguez-Chía, 2011). In this paper, we study the natural combination of Díaz-Báñez et al. (2012b) and Espejo and Rodríguez-Chía (2011) (where the total weighted transportation time is the function to minimize), and Díaz-Báñez et al. (2012a) (where the highway can have any arbitrary length).

1.4. Results

We first show that there exists an optimal solution of the FFL-problem in which the facility point f is located on h, and the highway h has infinite length (i.e., it is a line). We also characterize the way in which demand points travel to such a solution. This discretization on the shortest path shapes allows us to simplify the expression of $d_h(p,f)$ and to obtain a clear expression of the objective function $\Phi(f,h)$. Using geometric observations, we discretize the search space, and give an $O(n^3)$-time algorithm to solve the FFL-problem.

As a surprising result, our characterization of the search space depends on the highway's speed. To the best of our knowledge, this is the first time in the study of this kind of location problems that a detailed characterization of the solutions, depending on the highway's speed, is given. We conclude by presenting several examples that justify our characterization.

1.5. Outline

The discretization on the shapes of the shortest paths from the demand points to the facility, and properties of the objective function, are stated in Section 2 and in Section 3, respectively. In Section 4 we show how the search space of optimal solutions can be reduced. In Section 5 the algorithm to solve the FFL-problem is presented and in Section 6 we give the refinement of it. In Section 7, the examples are presented. Finally, in Section 8, we present the conclusions and further research.

2. Discretization of the shortest paths

Any solution to our problem will be encoded by a pair of elements (f,h), where f is the facility point and h is the highway. Given f and h, we say that a demand point p does not use h (or goes directly to f) if $d_h(p,f)$ is equal to $\|p-f\|_1$. Otherwise we say that p uses h. Given a point u of the plane, let x_u and y_u denote the x- and y-coordinates of u, respectively.

Observe from Eqs. (1) and (2) that there always exists an optimal solution (f,h) of the FFL-problem in which the length of h is infinite. We then assume from this point forward that every solution satisfies that the highway is a straight line.[5]

Lemma 1. *There exists an optimal solution of the FFL-problem in which the facility point is located on the highway.*

Proof. Let (f,h) denote an optimal solution of the FFL-problem and suppose that f does not belong to h. Let S_h denote the set of demand points using h. For every demand point $p \in S_h$, let $q_1 := q_1(p)$ and $q_2 := q_2(p)$ the enter and exit point on h, respectively, of the shortest path connecting p and f. The shortest path connecting p and f decomposes into three parts: an L_1 path connecting p and q_1, a segment of h connecting q_1 and q_2, and an L_1 path connecting q_2 to f. By definition, we have $d_h(p,f) = \|p-q_1\|_1 + \|q_1-q_2\|_2/v + \|q_2-f\|_1$.

Let h' be the line containing f that is parallel to h. Let $q_3 := q_3(p)$ be the point $q_1 + (f - q_2) = f + (q_1 - q_2)$. Observe that, by construction, point $q_3(p)$ is in h' (since f is in h', and we are moving using a director vector of h). We claim that walking from p to q_3, and then using h' from q_3 to f gives a path whose travel time is equal to the one that uses h, see Fig. 1. First notice that the segment of endpoints q_3 and f is included in h', thus the new path is indeed feasible with the alternate highway location. Moreover, by definition of q_3, we have $\|p-q_3\|_1 = \|p-q_1\|_1 + \|q_2-f\|_1$, and $\|f-q_3\|_2 = \|q_1-q_2\|_2$. Thus, we conclude that h' is an alternative highway location in which travel times of any point $p \in S_h$ do not increase. Clearly, the travel time of points in $S \setminus S_h$ cannot increase either, thus (f,h') must also be an optimal solution of the FFL-problem. □

We also note that results similar to Lemma 1 for several variations of this problem, stating that the facility point belongs to the corresponding highway, have been found by Espejo and

[5] Details of how to shorten the highway to a segment will be given in Section 5.