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Discrete Optimization

# Hop constrained Steiner trees with multiple root nodes

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ABSTRACT

We consider a network design problem that generalizes the hop and diameter constrained Steiner tree problem as follows: Given an edge-weighted undirected graph with two disjoint subsets representing *roots* and *terminals*, find a minimum-weight subtree that spans all the roots and terminals so that the number of hops between each *relevant node* and an arbitrary root does not exceed a given hop limit $H$. The set of relevant nodes may be equal to the set of terminals, or to the union of terminals and root nodes. This article proposes integer linear programming models utilizing one layered graph for each root node. Different possibilities to relate solutions on each of the layered graphs as well as additional strengthening inequalities are then discussed. Furthermore, theoretical comparisons between these models and to previously proposed flow- and path-based formulations are given. To solve the problem to optimality, we implement branch-and-cut algorithms for the layered graph formulations. Our computational study shows their clear advantages over previously existing approaches.



## 1. Introduction

Quality-of-service aspects are among the major issues when designing modern telecommunication networks and in particular bounding the maximum overall delay of each relevant communication path is important. It is widely accepted that in many applications the delay along some connection mainly depends on the number of intermediate routers, i.e., hops, and that restricting the maximum length of each established path by some predefined threshold limits the probability of failures (see, e.g., Dahl, Gouveia, & Requejo, 2006; Salama, Reeves, & Viniotis, 1996). Furthermore, whenever redundancy is not of major importance it is usually desired that the final network has tree structure in order to ensure unique communication paths and to reduce the maintenance effort, cf. Salama (1996) and Salama et al. (1996).

The literature contains many works dedicated to two problems that fit into this framework, namely the "centralized" hop-constrained minimum spanning/Steiner tree problem (HMSTP/HMStTP), see, e.g., Dahl et al. (2006), Gouveia (1995), Gouveia and Requejo (2001), Gouveia, Paias, and Sharma (2011), Gouveia, Simonetti, and Uchoa (2011), and Voß (1999) and the references therein, and the "decentralized" diameter-constrained minimum spanning/Steiner tree problem (DMSTP/DMStTP), see, e.g., Achuthan, Caccetta, Caccetta, and Geelen (1994), Gouveia and Magnanti (2003), Gouveia, Magnanti, and Requejo (2004, 2006), Gouveia et al. (2011), and Gruber (2009) and the references therein.

To define the HMSTP consider an undirected, edge-weighted graph $G = (V, E)$ with node set $V$, edge set $E$, a hop limit $H \in \mathbb{N}$, and one dedicated central node $r \in V$. The objective is to identify a minimum cost spanning tree such that the path between the root $r$ and any node $v \in V$ does consist of at most $H$ edges. For the Steiner variant (HMStTP) we are further given a set of terminals $T \subset V$ and the aim is to identify a minimum cost Steiner tree connecting all terminals such that the path between the root $r$ and any terminal node $t \in T$ does consist of at most $H$ edges. To define the DMSTP consider, as before, an undirected, edge-weighted graph. The objective is to identify a minimum cost spanning tree such that the path between any two nodes does consist of at most $D$ edges, for some given diameter limit $D \in \mathbb{N}$. Changes to the Steiner variant (DMStTP) are analogous to the hop-constrained problems.

However, several other tree problems with hop constraints appear to be of practical interest and one objective of this work is to propose a more general framework to contextualize these problems. In practice we may have multiple (e.g., replicated) central servers in which case each server communicates with a subset of terminals, and lengths of the corresponding communication paths are limited. One of the important *sparse mode multicast routing*

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protocols is based on *core-based trees* (CBTs) (Ballardie, Francis, & Crowcroft, 1993). In this protocol, a set of "core routers" is given, and they all multicast the information to a set of other *relevant nodes* (these correspond to receivers, that can be other routers or even end users). In classical multicast routing, each core router builds its own communication tree (also known as the *source tree architecture*), connecting the core router with the group of its relevant nodes. In the sparse mode multicast routing, however, it is required that the union of subtrees associated to the core routers builds a single tree. In this latter concept, also known as the *shared-tree architecture*, a common tree is built that connects all core routers and their relevant nodes, cf. Gossain, de Morais Cordeiro, and Carlos (2002), Salama (1996), and Salama et al. (1996). CBTs offer better scalability when compared to the source tree architecture and their main applications are for the Internet Protocol Television (IPTV) and in Mobile Digital Video Broadcasting-Handheld (DVB-H), see Minoli (2007). To ensure that the communication delays are not too high and also to ensure a certain reliability of the network, additional *hop constraints* may be imposed along the communication paths, e.g., between each server–receiver pair, cf. Dahl et al. (2006).

In this paper, we provide a *new generic mathematical model* for the application described above. The problem is called the *Hop Constrained Minimum Steiner Tree Problem with Multiple Root nodes* (HSTPMR) problem. We are given an undirected graph $G = (V, E)$, with node set $V$, edge set $E$, edge costs $c_e \geqslant 0$, for all $e \in E$, and a hop limit $H \in \mathbb{N}$. The node set $V$ contains two disjoint subsets: root nodes $R$, $|R| \geqslant 1$, and terminal nodes $T \subseteq V \setminus R$. Furthermore, we are given a set $T' \subseteq T \cup R$ of *relevant nodes* for which hop limits to all root nodes need to be considered.

A solution to the HSTPMR is a Steiner tree $G' = (V', E')$ spanning all root and terminal nodes, i.e., $R \cup T \subseteq V'$, such that the hop constraints are met for all relevant nodes $v \in T'$. More precisely, for each relevant node $t \in T'$ and each root $r \in R$, the unique path between $t$ and $r$ can contain at most $H$ edges. The objective is to find a feasible subtree yielding minimum total edge costs. If $T \cup R = V$, the solution will be a spanning tree of $G$.

In this study we consider two particular cases of this new framework which as far as we know have not been studied before (with exception to the introductory work in Gouveia, Leitner, & Ljubić (2012a)): (a) $T' = T \cup R$ and (b) $T' = T$. In the first case, delay bounds between roots have to be taken into consideration (e.g., when roots model replica servers) and in the second case delays between roots are not critical (e.g., when services by different providers are offered to terminals). An illustrative instance of the HSTPMR with two roots and three terminals is given in Fig. 1a, while Fig. 1b and c depict solutions to this instance for $T' = T \cup R$ and $T' = T$, respectively, assuming that $H = 3$. Notice that one could generalize this problem even further by introducing subsets of roots and hop limits that would depend on each node from $T'$.

However, the two cases already present different characteristics that strongly affect the corresponding models. For the case $T' = T \cup R$, it is easy to see that the hop-constrained arborescences associated to each root span the same set of nodes and the same set of undirected edges. This property is useful to strengthen the models that will be proposed in the next subsection. Unfortunately, this property may not be satisfied in the case $T' = T$ since the maximum distance between any two roots may exceed $H$. In fact as can be deduced from Fig. 1c, the subtree obtained from undirecting the arcs of the hop-constrained arborescence associated to root 0 does not coincide with the subtree obtained from undirecting the arcs of the hop-constrained arborescence associated to root 1. Thus, many of the model enhancements valid for the case $T' = T \cup R$ that we will discuss below, will not be valid for $T' = T$. The following results, however, provide an upper bound on the maximum distance between any two roots.

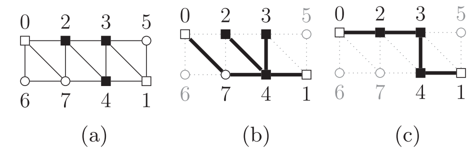


**Fig. 1.** (a) An illustrative instance with $R = \{0, 1\}$, $T = \{2, 3, 4\}$, and potential Steiner nodes $S = \{5, 6, 7\}$. (b) A feasible solution for $T' = T \cup R$ and $H = 3$. (c) A feasible solution for $T' = T$ and $H = 3$.

**Lemma 1.** *Let $G' = (V', E')$ be a feasible solution to an instance of the HSTPMR with $T' = T$ and let $d(u, v)$ denote the distance between two nodes $u, v \in V'$ in $G'$. Then, the maximum distance between any pair of root nodes in $G'$ does not exceed $2H - \ell$ where $\ell$ is the maximum distance between any two terminal nodes in $G'$, i.e., $\ell = \max_{u,v \in T} d(u, v)$.*

**Proof.** If there is a single terminal, two roots can be each at distance $H$ from it, which gives the maximum distance of $2H$. Assume that $|T| \geqslant 2$, let $t_1$ and $t_2$ be two terminals at maximum distance and let $P = (t_1 = v_0, v_1, \ldots, v_\ell = t_2)$ ($v_i \in V'$ for $0 \leqslant i \leqslant \ell$, and $\{v_i, v_{i+1}\} \in E'$ for $0 \leqslant i \leqslant \ell - 1$) denote the path between $t_1$ and $t_2$ in $G'$. Furthermore, let $r \in R$ be an arbitrary root and $v_j \in P, 0 \leqslant j \leqslant \ell$, be the node from $P$ such that the path between $r$ and $v_j$ is edge disjoint to $P$. Since, the maximum distance between a terminal and a root node may not exceed $H$, we have

$$d(r, v_j) \leqslant \begin{cases} H - \ell + j & \text{if } j \leqslant \ell/2 \\ H - j & \text{if } j \geqslant \ell/2 \end{cases}$$

Now let $s \in R$ be another root and $v_k \in P, 0 \leqslant k \leqslant \ell$ again be the node from $P$ such that the path between $s$ and $v_k$ is edge disjoint to $P$. Without loss of generality we assume that $j \leqslant k$. Then, by case distinction it is easy to see that

$$d(r, s) = d(r, v_j) + d(v_j, v_k) + d(s, v_k) \leqslant 2H - \ell$$

holds and that this bound can be tight. □

The next corollary immediately follows from Lemma 1.

**Corollary 1.** *Let $diam(T)$ be the minimum diameter of a subtree of $G$ spanning all nodes from $T$. Then, for any feasible solution $G' = (V', E')$ to an instance of the HSTPMR on $G$ with $T' = T$ the maximum distance between any pair of root nodes in $G'$ does not exceed $H'$, where $H' = 2H - diam(T)$.*

Notice that $H'$ can be calculated in polynomial time: It suffices to run breadth-first-search starting from each $t \in T$ until all remaining terminals are reached. The subtree with the smallest diameter obtained gives us the value of $diam(T)$. As we will show in Section 3.4, this corollary allows us to provide modified models, where many of the enhancements directly valid for the case $T' = T \cup R$ apply. The drawback is that these modified models use many more variables and constraints than the original model without the enhancements.

*Our Contribution.* In this paper, besides introducing the general and new problem we present three kinds of results: (a) Complexity: We analyze special cases in which the HSTPMR can be reduced to previously studied network design problems, identify special polynomial cases, show that the problem is NP-hard in general, and that one cannot guarantee to find an approximation ratio better than $\Theta(\log |V|)$ unless P = NP. (b) Mixed integer programming (MIP) models: We discuss layered graph reformulations, present strengthening valid inequalities and show that the obtained models theoretically dominate flow- and path-based models studied in Gouveia et al. (2012a). (c) Computational results: Branch-and-cut algorithms are developed for layered graph models and computa-