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Decision Support

Fuzzy analytic hierarchy process: Fallacy of the popular methods

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ABSTRACT

The fuzzy Analytic Hierarchy Process (fuzzy AHP) is a very popular decision making method and literally thousands of papers have been published about it. However, we find the basic logic of this approach has problems. From its methodology, the definition and operational rules of fuzzy numbers not only oppose the main logic of fuzzy set theory, but also oppose the basic principles of the AHP. In dealing with the outcomes, fuzzy AHP does not give a generally accepted method to rank fuzzy numbers and a way to check the validity of the results. Besides, we discuss the validity of the Analytic Hierarchy/Network Process (AHP/ANP) in complex and uncertain environments and find that fuzzy ANP is a false proposition because there is no fuzzy priority in the super matrix which provides the basis for the ANP. Although fuzzy AHP has been applied in many cases and cited hundreds of times, we hoped that those who use fuzzy AHP would understand the problems associated with this method.

1. Introduction

The Analytic Hierarchy Process (AHP), a theory for the measurement of intangibles side by side with tangibles, has been extensively used as a multi-criteria decision making (MCDM) tool and as a way for computing priorities (Saaty, 2010). Some researchers, however, have thought that it is better to make the judgments fuzzy when dealing with decisions in complex and uncertain environments. They think 1–9 fundamental scale of the AHP is a scale of crisp numbers and thus resolved to fuzzify those numbers. They assume that by using fuzzy judgments (e.g. triangular; trapezoidal; interval and fuzzy numbers) instead of the usual 1–9 fundamental scale in making pairwise comparisons, they offer a better decision model. The fuzzy judgments idea is widely used in AHP applications, as evidenced by more than one thousand journal publications, in which the authors take for granted that they need to fuzzify the judgments to make applications with greater validity.

As we know, there is a large number of papers on applications of fuzzy AHP, so Thomas L. Saaty, the architect of the AHP, has paid close attention, and written three papers (Saaty, 2006, Saaty and Liem, 2007, 2010) noting that: the fundamental scale of the AHP is already fuzzy, and has shown through examples that the fuzzy approach does not yield better results than the original eigenvector procedure in the AHP. In addition to Saaty and Tran, Wang, Luo, and Hua (2008) point out with numerical examples that the extent analysis method of triangular fuzzy AHP proposed by Chang (1996) yields zero-weight for some criteria or alternatives. Zhu, Jennifer, and Yang (2012) provide a theoretical proof why Chang's method (1996) yields zero-weight and expose other problems such as poor robustness, unreasonable priorities and information loss. Zhu (2013) proves that the outcome of triangular fuzzy AHP (Laarhoven & Pedrycz, 1983) is less valid in practice than the eigenvector priority outcome of the AHP. Dubois (2011) shows that in the fuzzy AHP the reciprocal condition cannot be guaranteed and fuzzy eigenvalues are hard to define in a rigorous way.

However, the above researchers are concern more about specific fuzzy AHP approaches. They do not give a general critique on the basic logic of fuzzy AHP. In this paper we give general comments on the basic mathematical logic of fuzzy AHP which include the relationships between fuzzy AHP and fuzzy set theory, the definition and operational rules of fuzzy numbers, and the efficiency of fuzzy judgments. On the basis of these analyses, we expect to help decision makers get to know the flaws in fuzzy AHP approaches.

Because the fuzzy AHP ranges over a number of topics, a point-by-point response to these arguments would become tedious, and detracts from the major point of our observations. Thus we choose three traditional research papers Laarhoven and Pedrycz (1983), Buckley (1985b) and Chang (1996) because they have a wide influence on the theories and applications of fuzzy AHP evidenced by a great number of citations in Google Scholar 1287, then 1238, and 1127 respectively. Of course, there are some literatures about fuzzy AHP which do not belong to the three papers we listed above and they can be divided into two categories: one is fuzzy preference relation, and the other is interval fuzzy number. The former uses fuzzy binary relations satisfying reciprocity and max–min

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transitivity (Tanino, 1984). This method uses elements in [0,1] to define a fuzzy binary relation, which is different from the three traditional and influential fuzzy AHP papers. Therefore, we do not consider this method. The latter is identical with the three traditional papers in mathematical logic. The difference is that they use different methods to derive the fuzzy weight. For example, Cheng (1996) changes fuzzy matrix into a crisp matrix and uses a-cuts to compute the eigenvector (weight vector) from the crisp matrix. Fedrizzi and Pereina (1995) discuss a way of finding fuzzy λ_{max}, where λ_{max} is the largest eigenvalue of a fuzzy matrix. Thus, the latter is similar to the three traditional and influential fuzzy AHP papers, and we consider it in our manuscript.

This paper is organized as follows: in Section 2 we review the AHP, fuzzy set theory and the basic logic of fuzzy AHP. The major thrust of this paper begins in Section 3 where we argue that fuzzy AHP is not really a fuzzy set theory because of the improper definition of fuzzy number and also because no membership grade is used in fuzzy AHP. In Section 4, we argue that fuzzy AHP lacks mathematical validity as it violates the basic principles of the AHP including the reciprocal and continuity axioms and the operational rule of consistency. Section 5 states that fuzzy AHP as a decision making approach lacks a generally accepted method to rank fuzzy numbers and a valid method to check the consistency index. In Section 6, we discuss the validity of the AHP and show why the AHP and its generalization the ANP without fuzziness are more effective than fuzzy AHP. In this section, we also find that fuzzy ANP is a false proposition because there is no fuzzy priority in the super matrix which provides the bases for the ANP. We conclude the paper in Section 7 by summarizing our findings and conclusions.

2. Preliminaries

In this section, we gave background materials on the AHP, fuzzy set theory and the mathematical logic of fuzzy AHP.

2.1. AHP

Saaty (1972, 1977, 1980) developed the Analytic Hierarchy Process (AHP) on the basis of four relatively simple axioms: reciprocal judgments, homogeneous comparisons, hierarchic network structures and syntheses and finally meeting desirable expectations (Saaty, 1986). Up to now, AHP has been widely used in solving many complicated decision-making problems, such as human acts (Saaty & Shang, 2011) and consistency test (Ergu, Kou, Peng, & Shi, 2011; Grošelj & Zadnik, 2012).

To derive the priorities for reasons of transitivity of inconsistency, AHP must rely on the principal eigenvector to estimate the priorities. The eigenvalue method (EM) is the solution of the eigenvalue problem:

$$Aw = \lambda_{max} w \tag{1}$$

where λ_{max} is the largest or principal eigenvalue of the pairwise comparison matrix A and w is the corresponding principal eigenvector.

Let a_{ij} represent the relative dominance of A_i over A_j. Let the matrix corresponding to the reciprocal pairwise relation be denoted by $A = (a_{ij})$ with the reciprocal property $a_{ji} = 1/a_{ij}$. The relative dominance of A_i over A_j along paths of length k is given by:

$$\frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}}$$

where $a_{ij}^{(k)}$ is the (i,j) entry of the kth power of the matrix (a_{ij}). The total dominance $w(A_i)$ of alternative i over all other alternatives along paths of all lengths is given by the infinite series:

$$w(A_i) = \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}} \tag{2}$$

which coincides with the Cesaro sum:

$$\lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}}$$

2.2. Fuzzy set theory

When Bellman and Zadeh (1970) introduced fuzzy set into decision-making, they gave a precise definition of a fuzzy set as follows:

Definition. Let $X = \{x\}$ denote a collection of objects (points) denoted generically by x. Then a fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, \mu_A(x))\}, \quad x \in X \tag{3}$$

where $\mu_A(x)$ is termed the *grade of membership of x in A*, and μ_A: $X \to M$ is a function from X to a space M called the *membership space*. When M contains only two points, 0 and 1, A is non-fuzzy and its membership function becomes identical with the characteristic function of a non-fuzzy set.

Example. Let $X = \{1, 2, \ldots\}$ be the collection of non-negative integers. In this space, the fuzzy set A of "several objects" may be defined (subjectively) as the collection of ordered pairs:

$$A = \{(3, 0.6), (4, 0.8), (5, 1.0), (6, 1.0), (7, 0.8), (8, 0.6)\}$$

2.3. The mathematical logic of fuzzy AHP

We review the mathematical logic of fuzzy AHP from the three earliest researchers: Laarhoven and Pedrycz (1983), Buckley (1985b) and Chang (1996) because these three papers have a wide influence on the theories and applications of fuzzy AHP evidenced by a great number of citations in Google Scholar 1287, then 1238, and 1127 respectively.

The earliest work on fuzzy AHP is proposed by Laarhoven and Pedrycz (1983) which uses fuzzy judgments with triangular fuzzy numbers. Later, Buckley (1985b) extended Laarhoven and Pedrycz's method by using trapezoidal fuzzy numbers to determine fuzzy priorities. Nearly ten years later, Chang (1996) gave an extension method of Laarhoven–Pedrycz's to obtain a *crisp priority vector* instead of fuzzy priority vector. Now, we demonstrate the operational rules of fuzzy AHP.

A. Definition of triangular fuzzy AHP (Chang, 1996; Laarhoven & Pedrycz, 1983)

For a fuzzy number M on $(\mathbb{R} = (-\infty, +\infty))$ to be a triangular fuzzy number, its membership function $\mu_M: \mathbb{R} \to [0,1]$ must be equal to:

$$\mu_M(x) = \begin{cases} \frac{1}{m-l}x - \frac{l}{m-l}, & x \in [l, m] \\ \frac{1}{m-u}x - \frac{u}{m-u}, & x \in [m, u] \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

with $l \leqslant m \leqslant u$, l and u stand for the lower and upper value of the support of M, respectively, and m for the modal value. The triangular fuzzy number, as given by Eqs. (4), will be denoted by (l, m, u). The support of M is the set of elements $\{x \in \mathbb{R} | l < x < u\}$. The operations on triangular fuzzy numbers will be shown as follows: