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A combined scalarizing method for multiobjective programming problems

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ABSTRACT

In this paper, a new general scalarization technique for solving multiobjective optimization problems is presented. After studying the properties of this formulation, two problems as special cases of this general formula are considered. It is shown that some well-known methods such as the weighted sum method, the ϵ-constraint method, the Benson method, the hybrid method and the elastic ϵ-constraint method can be subsumed under these two problems. Then, considering approximate solutions, some relationships between ε-(weakly, properly) efficient points of a general (without any convexity assumption) multiobjective optimization problem and ϵ-optimal solutions of the introduced scalarized problem are achieved.

1. Introduction

One part of mathematical programming is multiobjective optimization programming when the conflicting objective functions must be minimized over a feasible set of decisions. In many areas in engineering, economics, and science new developments are only possible by the application of multiobjective optimization problems (MOPs) and related methods. There are many recent publications on applications of MOPs (Ehrgott, Klamroth, & Schwehm, 2004; Hillermeier & Jahn, 2005; Hutterer & Jahn, 2003; Jahn, 2004; Steuer & Na, 2003), and many others. Various monographs collected many results in theory and methodology (Ehrgott, 2000; Eichfelder, 2009; Ruzika & Wiecek, 2005), or provided a comprehensive review of methods (Marler & Arora, 2003). For solving MOPs, there are a number of methods and algorithms which are classified according to participation of the decision maker in the solution process (Hwang & Masud, 1979). The traditional and common approach for solving MOPs is a reformulation as a parameter scalar optimization problem. In other words, they are most commonly solved indirectly by using conventional (single-objective) optimization techniques by the aid of scalarization. In general, scalarization means the replacement of a vector optimization problem by a suitable scalar optimization problem which is an optimization problem with a real valued objective function. Since the scalar optimization theory has been widely developed, scalarization turns out to be of great importance for the vector optimization theory, as it is done in the well known weighted sum method (Geoffrion, 1968; Marler & Arora, 2010), the ϵ-constraint method (Chankong & Haimes, 1983; Mavrotas, 2009), the hybrid method (Guddat, Guerra, Tammer, & Wendler, 1985; Huang & Yang, 2002), the Benson method (Benson, 1998), the normal boundary intersection method (Das & Dennis, 1998), and so on. For a survey on the scalarizing technique, the reader is referred to Ehrgott and Wiecek (2005). Our focus in this paper is based on the main idea of the elastic ε-constraint method introduced by Ehrgott and Ruzika in Ehrgott and Ruzika (2008). Since the ε-constraint method has no result about properly efficient solutions, Ehrgott and Ruzika have presented two modifications of the ε-constraint method to remedy this weakness. We use their strategy to constitute a general form. We show that the weighted sum method, the ϵ-constraint method, the Benson method, the hybrid method and the elastic ϵ-constraint method can be seen as special cases of our problem. Then, we prove some necessary and sufficient conditions for (weakly, properly) efficient points of a general MOP via optimal solutions of the presented scalarized problem. Researchers have tried to present general formulations for multiobjective optimization problems. For example, Luque, Ruiz, and Miettinen (2011), Romero (2001) and Ruiz, Luque, and Miettinen (2012), introduced a general formulation for several interactive methods. Their general formulation can accomodate some well-known interactive methods. Our formulation in this paper is not for interactive methods and so, is different from the formulation in Luque et al. (2011) and Ruiz et al. (2012). It should be mentioned that there exist several publications about properly efficient solutions (Chankong & Haimes, 1983; Huang & Yang, 2002), and many others, which use terms of stability of the scalarized problem or

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the K.K.T multipliers. However, our results on proper efficiency are more direct.

On the other hand, the importance of approximation solutions for MOPs in recent decades motivated us to investigate ε-efficient solutions. The first notion of approximation was suggested by Kutateladze (1979) and extended by Loridan (1984). White (1986) investigated six kinds of ε-approximate efficient solutions. Many authors studied the properties of this kind of solution. Some necessary and sufficient conditions for ε-(weak) efficiency can be found in Dutta and Vetrivel (2001), Gutierrez, Jimenez, and Novo (2006, 2007) and others. Engau and Wiecek (2007) investigated scalarization approaches to generate ε-efficient solutions of MOPs. Since our presented problems are extensions of methods in Engau and Wiecek (2007), the results in the current paper are extension of those of special cases in Engau and Wiecek (2007). Also, one of the most important notions in multiobjective optimization theory is proper efficiency introduced by Li and Wang (1998). Liu (1999) derived some necessary and sufficient conditions for ε-proper efficient solutions of convex MOPs. See also Beldiman, Panaitescu, and Dogaru (2008), Gao, Yang, and Lee (2010), Gao, Yang, and Teo (2011). The methods considered in Engau and Wiecek (2007) have no result on ε-proper efficiency. So, Ghaznavi and Khorram (2011) and Ghaznavi, Khorram, and Soleimani-Damaneh (2012), using the elastic ε-constraint method, provided some necessary and sufficient conditions for ε-(weak, proper) efficiency. Since our problem is a general form and the elastic ε-constraint method is a special case of that, the obtained results extend the results obtained in Ghaznavi and Khorram (2011), Ghaznavi et al. (2012) and Engau and Wiecek (2007). It is worth mentioning that the obtained results are general and we do not assume any convexity assumption.

The outline of this article is as follows: in Section 2, we provide preliminaries and basic definitions. In Section 3, we present the general formulation and study some properties of this formula. In Sections 4 and 5, two problems are presented which are special cases of the general formula presented in Section 3. Section 6 is devoted to the necessary and sufficient conditions to obtain ε-(weakly, properly) efficient solutions in three subsections. The conclusions are derived in Section 7.

2. Preliminaries and basic definitions

In this paper, optimization of the multiple objective problem is studied as follows:

$$\begin{aligned} \min f(x) &= (f_1(x), f_2(x), \ldots, f_p(x)) \\ g_i(x) &\leqslant 0, \quad i = 1,2,\ldots,m \\ h_k(x) &= 0, \quad k = 1,2,\ldots,\acute{m} \end{aligned} \tag{2.1}$$

where

$$f_j, g_i, h_k : \ \Omega \subset R^n \to R, \quad \forall j, i, k,$$

and $\Omega \neq \emptyset$. Here, we show all the feasible points by X. In other words,

$$X = \{x \in \Omega | g_i(x) \leqslant 0, h_k(x) = 0, \forall i, k\}.$$

Now, the following definitions are presented to determine efficient solutions of the MOP.

Definition 2.1. A feasible solution $x^* \in X$ of the MOP is called

(1) Efficient optimal solution if there does not exist another $x \in X$ such that

$$f_j(x) \leqslant f_j(x^*) \quad \text{for all} \quad j = 1,2,\ldots,p \quad \text{and } f(x) \neq f(x^*).$$

(2) Weakly efficient solution if there is no $x \in X$ such that

$$f_j(x) < f_j(x^*); \quad j = 1,2,\ldots,p.$$

(3) Strictly efficient solution if there does not exist another feasible solution $x \neq x^*$ such that

$$f_j(x) \leqslant f_j(x^*); \quad j = 1,2,\ldots,p.$$

Let $X_E(X_{wE}, X_{sE})$ be the set of efficient(weakly, strictly efficient) solutions. If x^* is an efficient (weakly efficient) solution, $f(x^*)$ is called a nondominated (weakly nondominated) point. The set of nondominated (weakly nondominated) points is denoted by $Y_N(Y_{wN})$. In other words, $Y_N := f(X_E)(Y_{wN} = f(X_{wE}))$.

We assume throughout this paper that $Y = f(X)$ is bounded and that X_E is nonempty. This is guaranteed, e.g. if X is compact and f_i are continuous (see Ehrgott, 2000).

Throughout this paper, we use the following notations:

- $R^p_> := \{y \in R^p | y_i > 0, i = 1,2,\ldots,p\}$.
- $R^p_\geqslant := \{y \in R^p | y_i \geqslant 0, i = 1,2,\ldots,p\} \setminus \{0\}$.
- $R^p_\geqslant := \{y \in R^p | y_i \geqslant 0, i = 1,2,\ldots,p\}$.

On the other hand, there exists a well-known kind of efficient points which are named properly efficient solutions. Properly efficient points are those efficient solutions that have bounded trade-offs between the objectives. There are some definitions for proper efficiency given by Benson (1979), Borwein (1977) and Hartley (1978) and others. Here we use the definition of proper efficiency in the sense of Geoffrion (1968).

Definition 2.2. A feasible solution $\hat{x} \in X$ is called properly efficient in Geoffrions's sense, if it is efficient and if there is a real number $M > 0$ such that for all i and $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index j such that $f_j(\hat{x}) < f_j(x)$ and

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} < M.$$

The set of properly efficient solutions is denoted by X_{pE}.

ε-(weakly) efficient solutions of MOP (2.1) are defined as follows Loridan (1984):

Definition 2.3. Take into consideration MOP (2.1). Let $\varepsilon \in R_{\geqq}$. A feasible point $\hat{x} \in X$ is called:

(1) ε-Weakly efficient if there is no other $x \in X$ such that $f(x) < f(\hat{x}) - \varepsilon$.
(2) ε-Efficient if there is no other $x \in X$ such that $f(x) \leqslant f(\hat{x}) - \varepsilon$.

Definition 2.4 (Li and Wang, 1998). A feasible point $\hat{x} \in X$ is called ε-properly efficient point of problem (2.1), if it is ε-efficient and there is a real positive number $M > 0$ such that for all $i \in \{1,2,\ldots,p\}$ and $x \in X$ satisfying $f_i(x) < f_i(\hat{x}) - \varepsilon_i$, there exists an index $j \in \{1,2,\ldots,p\}$ such that $f_j(\hat{x}) - \varepsilon_j < f_j(x)$ and

$$\frac{f_i(\hat{x}) - f_i(x) - \varepsilon_i}{f_j(x) - f_j(\hat{x}) + \varepsilon_j} < M.$$

The set of all ε-weakly efficient, ε-efficient and ε-properly efficient solutions of an MOP will be indicated by $X_{\varepsilon WE}$, $X_{\varepsilon E}$ and $X_{\varepsilon PE}$, respectively. Notice that for $\varepsilon = 0$, ε-weak efficiency, ε-efficiency and ε-properly efficiency collapse in the usual definition of weak efficiency, efficiency, (Definition 2.1) and properly efficiency (Definition 2.2).

Remark 2.5. Obviously, $X_{\varepsilon PE} \subseteq X_{\varepsilon E} \subseteq X_{\varepsilon WE}$.

The customary approach to solve a given MOP is to formulate a single objective program (SOP) associated with it. Let us consider an SOP as follows: