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Decision Support

Achievable hierarchies in voting games with abstention

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ABSTRACT

It is well known that he influence relation orders the voters the same way as the classical Banzhaf and Shapley–Shubik indices do when they are extended to the voting games with abstention (VGA) in the class of complete games. Moreover, all hierarchies for the influence relation are achievable in the class of complete VGA. The aim of this paper is twofold. Firstly, we show that all hierarchies are achievable in a subclass of weighted VGA, the class of weighted games for which a single weight is assigned to voters. Secondly, we conduct a partial study of achievable hierarchies within the subclass of H-complete games, that is, complete games under stronger versions of influence relation.

1. Introduction

In a committee, a voting rule specifies the decision-making procedure, that is, when a proposal is to be accepted or rejected depending on the resulting vote configuration. The vote configuration itself depends on different options offered to committee members. A very huge class of voting rules studied in the literature deals with simple games. In such voting models, any voter either votes for or against the proposal. If a voter does not favor a proposal, then he is considered to be against, that is, an abstention if any, is treated as a vote against the proposal. It is well known however that many decision-making processes including relative majority, vote in the United Nations Senate cannot be fitted in such models.

To handle this shortcoming, models of voting games with abstention (VGA) were introduced (Rubinstein (1980) defined social decision systems, Felsenthal and Machover (1997, 1998) defined Ternary Voting Game), where each player is allowed three distinct votes but the outcome of the vote still has two options. An important but isolated earlier work on abstention can be found in Fishburn's book Fishburn (1973). More recently, Freixas and Zwicker (2003) extended VGAs to the so-called (j,k) games, a subclass of which are $(j,2)$ games in which voters have j possible ordered levels of approval in the input, thus partitioning the committee N into j coalitions, each attached to a winning or losing character in the output. Simple games constitute the class of $(2,2)$ simple games whereas VGAs constitute the class of $(3,2)$ simple games.

A fundamental question is the assessment of the influence of each voter to affect the outcome of a vote. Several power indices for simple games have so far been defined to capture the ability of the players to affect the voting outcome. The two most conspicuous representatives of this line of research are the Shapley–Shubik (SS) power index (Shapley & Shubik, 1954) and the Banzhaf and Coleman (BC) power indices (Banzhaf, 1965; Coleman, 1971) originally defined in voting rules modeled by simple games. In a quite distinct direction, the desirability relation (introduced by Isbell (1958) and extensively studied by Taylor (1995)) rank directly players according to their influence. Previous work by Felsenthal and Machover (1997) and Diffo Lambo and Moulen (2002) show that all these power theories are ordinally equivalent in the class of swap-robust simple games.

In order to capture the ordering of the influence held by the players in a game, the concept of hierarchy was introduced by Friedman, McGrath, and Parker (2006). For example, a five-player game G has hierarchy $(1,3,1)$ means that one player has less influence than all the others, one player has more, and the other three players have the same influence as each other, they are equivalent. From works by Friedman et al. (2006) and Freixas and Pons (2010) on hierarchies, it can be stated that given any complete pre-ordering defined on a finite set of more than 5 voters, it is possible to construct a simple game such that the pre-orderings induced by Shapley and Shubik (1954), and Banzhaf (1965) and Coleman

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(1971) power indices coincide with the given pre-ordering (when the number of voters is 5 or less, there are four non achievable hierarchies). These results hold under the condition that the game be modeled by a simple game. With respect to the construction of the hierarchies, recently, Bishnu and Roy (2012) have shown how to use minimal winning coalitions to extract the hierarchy of players.

This paper deals with voting games with abstention. The question of achievable hierarchies is relevant thanks to the fact that SS and BC indices have all been clearly generalized to VGA by Felsenthal and Machover (1997) while the Coleman index has been generalized to VGA by Freixas (2012). They have moreover been generalized to the most general model of (j,k) games by Freixas (2005a, 2005b). On the other hand, the desirability relation has been defined by Tchantcho, Diffo Lambo, Pongou, and Mbama Engoulou (2008) in terms of *I*-influence relation. These authors showed that the SS, BC and the *I*-influence relation are ordinally equivalent in the subclass of equitable swap-robust games. Recently, Parker (2012) showed that this ordinal equivalence holds in the whole class of swap-robust games.

With respect to hierarchies, we show in this paper that all hierarchies are achievable. More precisely, the four hierarchies cited above that were not achievable for simple games (when abstentions are not permitted to players) are achievable in a particular class of weighted games, the class of zero-centered strongly weighted VGA. This is a refinement of Parker's result Parker (2012). Weighted games as well as the characterization of this class of games were given by Freixas and Zwicker in Freixas and Zwicker (2003).

Freixas, Tchantcho, and Tedjeugang (2013) noticed some shortcoming in the *I*-influence. There are weighted games not being complete for the influence relation, something different to what occurs for simple games. They introduced several extensions of the desirability relation (see also Pongou, Tchantcho, & Diffo Lambo (2011)) by considering each condition in the definition of *I*-influence relation. A stronger form of *I*-completeness is *H*-completeness for which all the relations that intervene in the definition of completeness coincide. In this paper we also address the problem of achievable *H*-hierarchies. We show in particular that no strict hierarchy is achievable for games with 2 or 3 players. For games with 4 players, except the strict hierarchy for which we do not get any answer, all other hierarchies are achievable. For games with more than 5 players, the strict hierarchy is achievable in the class of *H*-complete $(3,2)$ games. Furthermore, unlike the subclass of hierarchies $(m,1,1)$ with $m \geqslant 2$, all other *H*-hierarchies are achievable.

Determining importance rankings is a significant issue in operational research. The study of ordinal preferences involves a variety of fields, including tournament theory, multiple criteria decision modeling (MCDM), and data envelopment analysis of qualitative data. As stated in the survey by Cook (2006), the notion of voter power or relative importance has been largely ignored in studies on ordinal ranking problems, although if a tangible estimate of voter importance exists, then these voters can be treated like criteria in an MCDM problem. The approach in our paper is useful in ranking voters in voting institutions where abstention is allowed as a third input. Examples of application of our results naturally apply to political institutions, but also in management enterprisers and even in reliability systems where voters are replaced by device components with three input levels. Examples in these different contexts can be found in: Levitin (2003), Obata and Ishii (2003), Alonso-Meijide, Bilbao, Casas-Méndez, and Fernández (2009), Sueyoshi, Shang, and Chiang (2009) or Freixas, Marciniak, and Pons (2012). The paper is organized as follows.

The technical background as well as some useful results are recalled in Section 2. In Section 3, we recall several notion of desirability for $(3,2)$ games and consider as well their completeness, their link with weightedness. In Section 4 we prove that all hierarchies induced by the influence relation are achievable in the particular class of weighted $(3,2)$ games, the class of zero-centered strongly weighted VGA. As for *H*-hierarchies a partial study is done in section 5 and a conclusion then ends the paper.

2. Preliminaries

An ordered 3-partition of N (set of voters or players) is a sequence $S=(S_1,S_2,S_3)$ of mutually disjoint subsets of N whose union is N. In S, S_1 stands for the set of yes voters, S_2 for abstainers and S_3 stands for no voters. We denote by 3^N the set of all ordered 3-partitions of N. For $S,S' \in 3^N$, we write $S \subset^3 S'$ if S can be transformed into S' by shifting one or more voters to higher levels of approval.

Definition 2.1. A $(3,2)$ game $G=(N,V)$ consists of a finite set N of voters together with a value function $V : 3^N \longrightarrow \{0,1\}$ such that for all ordered 3-partition S,S', if $S \subset^3 S'$ then $V(S)=1$ implies $V(S')=1$.

A 3-partition S such that $V(S)=1$ is said to be winning. A $(3,2)$ game can be defined by its set of winning 3-partitions, $W=\{S \in 3^N : V(S)=1\}$. In that case we denote the game by (N,W). In voting, it is often demanded that V be exhaustive, then from the monotonicity demanded to $V, V(\mathcal{N})=0$ and $V(\mathcal{M})=1$ where $\mathcal{N}$ and $\mathcal{M}$ are respectively the 3-partitions with $\mathcal{N}_3=N$ and $\mathcal{M}_1=N$. A special type of $(3,2)$ simple games is the class of anonymous or symmetric games which have been intensively studied in Freixas and Zwicker (2009). Anonymous $(3,2)$ games are games for which for all 3-partition S,S is winning if and only if for all permutations $\pi : N \to N, \pi(S)=(\pi(S_1),\pi(S_2),\pi(S_3))$ is winning.

Definition 2.2. In a $(3,2)$ game, a 3-partition S is said minimal winning if S is winning and for all $T \in 3^N$ such that $T \subset^3 S, T$ is losing. As well, S is said maximal losing if S is losing and for all $T \in 3^N$ such that $S \subset^3 T, T$ is winning.

Either the set of minimal winning 3-partitions or the set of maximal losing 3-partitions completely generates the game. Next, we introduce weighted $(3,2)$ games, which is a special type of weighted (j,k) games introduced in Freixas and Zwicker (2003).

Definition 2.3. Let $G=(N,V)$ be a $(3,2)$ game. A representation of G as a $(3,2)$ weighted game consists of a sequence $w=(w_1,w_2,w_3)$ of 3 weight functions, where $w_i : N \longrightarrow \mathbb{R}$ for each i with $w_1(p) \geqslant w_2(p) \geqslant w_3(p)$ for each $p \in N$, together with a real number Q so called quota such that for any 3-partition S,S is winning if and only if $w(S)=\sum_{i=1}^{3}\sum_{p\in S_i} w_i(p) \geqslant Q$.

According to this definition we can normalize, i.e. assign a zero weight, to any level of approval. Here we are mainly concerned with games with abstention for which we can normalize the weights at any of the three input levels, but we choose the "abstention" level which seems to be quite natural. If a null weight is assigned to abstainers, then a non-negative weight is assigned to "yes" voters and a non-positive weight to "no" voters. Thus, a weight[2] $w(p)=(w^+(p),0,w^-(p))$ with $w^+(p) \geqslant 0$ and $w^-(p) \leqslant 0$ is assigned to each $p \in N$. The only requirement for the threshold Q, if the $(3,2)$ game is demanded to be exhaustive, is

$$w(\mathcal{N}) = w(\emptyset,\emptyset,N) = \sum_{p\in N} w^-(p) < Q \leqslant \sum_{p\in N} w^+(p) = w(N,\emptyset,\emptyset)$$
$$= w(\mathcal{M}).$$

The previous definition can now be rewritten as follows.

[2] We are identifying w^+ with w_1, 0 with w_2 and w^- with w_3 in Definition 2.3.