ARTICLE IN PRESS

European Journal of Operational Research xxx (2013) xxx–xxx

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Enhancing data consistency in decision matrix: Adapting Hadamard model to mitigate judgment contradiction ☆

Gang Kou [a], Daji Ergu [b,c,*], Jennifer Shang [d]

[a] School of Business Administration, Southwestern University of Finance and Economics, Chengdu 610074, China
[b] School of Management and Economics, University of Electronic Science and Technology of China, Chengdu 610054, China
[c] Southwest University for Nationalities, Chengdu 610041, China
[d] Joseph M. Katz Graduate School of Business, University of Pittsburgh, USA

ARTICLE INFO

Article history:
Received 6 February 2013
Accepted 26 November 2013
Available online xxxx

Keywords:
Multiple criteria analysis
Hadamard product induced bias matrix
Pairwise comparison matrix
Cardinal and ordinal inconsistency
Graph theory

ABSTRACT

Cardinal and ordinal inconsistencies are important and popular research topics in the study of decision making with pair-wise comparison matrices (PCMs). Few of the currently-employed tactics are capable of simultaneously dealing with both cardinal and ordinal inconsistency issues in one model, and most are heavily dependent on the method chosen for weight (priorities) derivation or the obtained closest matrix by optimization method that may change many of the original values. In this paper, we propose a Hadamard product induced bias matrix model, which only requires the use of the data in the original matrix to identify and adjust the cardinally inconsistent element(s) in a PCM. Through graph theory and numerical examples, we show that the adapted Hadamard model is effective in identifying and eliminating the ordinal inconsistencies. Also, for the most inconsistent element identified in the matrix, we develop innovative methods to improve the consistency of a PCM. The proposed model is only dependent on the original matrix, is independent of the methods chosen to derive the priority vectors, and preserves most of the original information in matrix A since only the most inconsistent element(s) need(s) to be modified. Our method is much easier to implement than any of the existing models, and the values it recommends for replacement outperform those derived from the literature. It significantly enhances matrix consistency and improves the reliability of PCM decision making.

1. Introduction

Decision makers (DMs) often encounter complicated decision problems that involve multiple tangible and intangible conflicting criteria and alternatives (Forman & Gass, 2001; Raiffa & Keeney, 1976; Saaty, 1980; Tsetlin & Winkler, 2007; Vansnick, 1986). The intangible and subjective aspects of judgments associated with human factors need to be integrated into an open and flexible multi-criteria decision making model (Altuzarra, Moreno-Jiménez, & Salvador, 2010; Saaty, 1972). Due to the limitations of human capacity and DMs' experiences and knowledge, it is difficult to compare several criteria or alternatives simultaneously (Hu & Mehrotra, 2012; Saaty, 1986, 1994). However, it is relatively easy to determine the dominance of one alternative over the other with respect to a given criterion at a time. As such, the pairwise comparison method becomes an important tool for multi-criteria decision making. In this research, we focus on the pairwise comparison matrix with the goals of improving its consistency and enhancing its reliability in decision making.

The pairwise comparison technique, originated by Thurstone (1927), is widely employed to handle subjective and objective judgments in multi-criteria decision making (Herman & Koczkodaj, 1996; Saaty, 1980, 1994, 1996, 2006; Zhü, in press). All pair-compared results are arranged in a matrix $A = (a_{ij})_{n\times n}$, where $a_{ij} > 0$, $a_{ij} = 1/a_{ji}$ and $a_{ij} = a_{ik}a_{kj}$ for $i, j, k = 1,2,\ldots,n$, and popularly termed pairwise comparison matrix (PCM hereinafter) or judgment matrix in literature.

The values in PCM are provided by decision makers based on their judgment and expertise. The matrix may be inconsistent due to the complexity of the decision problem or the limits of DMs' capacities and skills. Thus, the consistency issue has been an ongoing and active research topic, and a number of models have been developed for it (e.g., Saaty, 1986; Barzilai, 1999; Cao, Leung, & Law, 2008; Li & Ma, 2007; Xu & Wei, 1999; Altuzarra et al., 2010; Ergu, Kou, Peng, & Shi, 2011; Grošelj & Zadnik Stirn, 2012; Siraj, Mikhailov, & Keane, 2012, and Liu, Zhang, & Zhang, in press).

* Corresponding author at: Southwest University for Nationalities, Chengdu 610041, China. Tel.: +86 18981708781.

E-mail addresses: kougang@yahoo.com (G. Kou), ergudaji@163.com (D. Ergu), shang@katz.pitt.edu (J. Shang).

0377-2217/$ - see front matter © 2013 The Authors. Published by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ejor.2013.11.035

The consistency issue in matrix $A = (a_{ij})_{n\times n}$ can be classified into two types: (1) cardinal inconsistency, e.g., if $A = mB$, $B = nC$, but $A \neq mnC$; and (2) ordinal inconsistency, e.g. if $A > B$, $B > C$, *but* $C \geqslant A$. More generally, if $a_{ij} = a_{ik}a_{kj}$ holds for all *i*, *j*, and *k*, then it is cardinal consistent, and matrix *A* is perfectly consistent; otherwise, it is cardinal inconsistent. Similarly, if $a_{ij} \geqslant 1$, $a_{jk} \geqslant 1$, and $a_{ik} \geqslant 1$, then it is ordinal consistent; otherwise, it is ordinal inconsistent.

With the above classification, we can group the methods for tackling the consistency issues into three groups. The first one deals with cardinal inconsistency, the second one deals with ordinal inconsistency, and the third one deals with both (see detailed review in Section 2). Although consistency ratio (CR) that relies on maximum eigenvalue is the commonly used index to test whether a PCM is consistent, we found that most consistency improvement methods require the priority weights of the PCM (see Sections 2 and EC.1). However, there are more than 20 methods (Choo & Wedley, 2004; Lin, 2007; Kou & Lin, in press) available for deriving the priority weights in a PCM. When the PCM is consistent, all methods will arrive at the same weights. However, if the PCM is inconsistent, the priority weights derived from different methods could differ significantly, and using such weights prevent us from correctly identifying the inconsistent elements and providing accurate estimates.

Through Monte-Carlo simulation, Siraj et al. (2012) show that a high percentage of consistent matrices (i.e., with CR < 0.1) are in fact ordinally inconsistent. They further prove that if the matrices are ordinally inconsistent, then different prioritization methods give different ordinal rankings. Yet, their model focuses solely on ordinal inconsistency. The Condition of Order Preservation (COP) proposed by Bana e Costa and Vansnick (2008) states that for alternatives A_i, A_j, A_k, and A_h, when A_i is preferred to A_j, and A_k is preferred to A_h, but the intensity of preference of A_i over A_j is stronger than that of A_k over A_h, then the priority weights ω not only satisfy $\omega_i > \omega_j$ and $\omega_k > \omega_h$ (preservation of order of preference) but also respect the relationship that $\omega_i/\omega_j > \omega_k/\omega_h$ (preservation of order of intensity of preference). They prove that some priority weights in the AHP do not satisfy the COP even when CR < 0.1. To tackle both types of inconsistencies in PCM, a requisite for a practicable model is that the proposed methodology does not depend on priority weights of the PCM.

In this paper, we tackle both cardinal and ordinal inconsistencies and aim to achieve two objectives. The first is to propose a method to identify cardinal inconsistency and to improve the consistency. Such a method has to be objective and independent of the priority-deriving methods chosen. To achieve such an objective, we transform the perfectly consistent condition $a_{ij} = a_{ik}a_{kj}$ into $a_{ik}a_{kj}a_{ji} = 1$. We discover that the Hadamard product operator in mathematics can be adapted to induce a bias matrix. We thus developed a Hadamard product induced bias matrix (HPIBM) to identify the inconsistent elements in a matrix efficiently and effectively. Different from the Hadamard product operator methods proposed by Saaty (2003), and Cao et al. (2008) (see Section 2.1), the proposed HPIBM only depends on the original data in PCM and is independent of the prioritization method chosen and weights derived.

The second goal of this paper is to extend the proposed HPIBM to identify and eliminate the ordinal inconsistency. Since three-way cycles are the basic forms of all ordinal inconsistencies, we further examine the forms of ordinally inconsistent judgments by combining the proposed HPIBM and graph theory. We found that the ordinal inconsistency can be readily identified by constructing the preference Boolean matrix and applying it to the proposed HPIBM.

The advantages of the HPIBM model we proposed in this research are fivefold:

(1) It is independent of the prioritization methods, since it is only based on the original numbers in PCM.
(2) It is easier than existing methods, as the most cardinally inconsistent elements can be identified quickly and effortlessly by observing the largest values in the induced bias matrix C.
(3) The inconsistent elements can be adjusted by the derived formula, and the consistency ratio can clearly be improved.
(4) It can also be used to identify the ordinal inconsistency.
(5) It is independent of the scale employed in the PCM and is suitable for any PCM whose entries are positive and reciprocal.

The remainder of this paper is organized as follows. Section 2 reviews the existing models for identifying inconsistency. In Section 3, we mathematically prove the validity of the proposed HPIBM model for cardinal inconsistency identification and develop procedures to improve the inconsistency. We then make use of graph theory in Section 4 to justify the application of HPIBM to ordinal inconsistency identification. Numerical examples are provided to illustrate and compare the proposed model in this section. The paper is summarized and concluded in Section 5.

2. Review of Inconsistency Identification models

Much attention has been paid to the inconsistency issues of PCM. Some researchers focus on cardinal inconsistency, while others concentrate on the ordinal inconsistency. Very few have addressed both inconsistency issues and none has tackled both cardinal and ordinal inconsistencies with one model. We review the inconsistency identification literature below.

2.1. Cardinal Inconsistency Identification Model

For a positive reciprocal decision making PCM $A = (a_{ij})_{n\times n}$, Harker (1987) proposed a formula based on the priority weights ω_i of positive matrix A; and the priority weights v_i of its transpose matrix A^T. He labeled the largest absolute value(s) as the most inconsistent element. Saaty (1980, 1994, 2003) believed the most inconsistent element can be identified by observing the maximum value in the absolute differences $B = [b_{ij}] = [|a_{ij} - \omega_i/\omega_j|]$ and the perturbation matrix $\varepsilon = A \circ W^T = [a_{ij}\omega_j/\omega_i]$, where ω_i and ω_j are the priority weights of matrix *A*, symbol "$\circ$" represents Hadamard product. Based on these models, Xu and Wei (1999) developed an auto-adaptive algorithm to improve the consistency. Similarly, from the perturbation matrix reviewed above, Cao et al. (2008) proposed an iterative algorithm to adjust the deviation matrix and improve the consistency ratio, which is based on the model $B = [\omega_i/\omega_j] \circ [a_{ij}/\omega_i/\omega_j]^{\gamma}$, where symbol "$\circ$" represents Hadamard product. Details of these models can be found in the e-companion, Section EC.1.

2.2. Ordinal Inconsistency Identification Model

Ali, Cook, and Kress (1986) defined the number of transitivity violations by checking whether $\omega_i > \omega_j$ and $a_{ij} < 1$. Gass (1998) used the methods from tournaments and graph theory to determine the number of three-way cycles in a PCM, and to detect these cycles through standard linear programming. Correspondingly, Kwiesielewicz and van Uden (2004) developed an algorithm to test the incongruity of the judgments in a PCM; while Birnbaum (2007) designed a statistical technique to test the intransitivity of preferences predicted by a lexicographic semi-order. Similarly, Diaye and Urdanivia (2009) found that the violation of the preference transitivity axiom significantly affects the violations of utility function