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Continuous optimization

A viscosity method with no spectral radius requirements for the split common fixed point problem

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ABSTRACT

This paper is concerned with an algorithmic solution to the split common fixed point problem in Hilbert spaces. Our method can be regarded as a variant of the "viscosity approximation method". Under very classical assumptions, we establish a strong convergence theorem with regard to involved operators belonging to the wide class of quasi-nonexpansive operators. In contrast with other related processes, our algorithm does not require any estimate of some spectral radius. The technique of analysis developed in this work is new and can be applied to many other fixed point iterations. Numerical experiments are also performed with regard to an inverse heat problem.

1. Introduction

Throughout this paper $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces endowed with inner products and induced norms denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively, while $\mathcal{H}$ refers to as any of these spaces. The fixed point set of any self-mapping U on $\mathcal{H}$ is denoted by $\mathrm{Fix}(U) := \{x \in \mathcal{H}; Ux = x\}$. The purpose of this work is to revisit the numerical approach to a solution of the *split common fixed point problem* (introduced in Censor & Segal (2009)), which is written

$$\text{find } x^* \in \mathrm{Fix}(T) \text{ such that } Ax^* \in \mathrm{Fix}(S), \tag{1.1}$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a (nonzero) bounded linear operator, $T : \mathcal{H}_1 \to \mathcal{H}_1$ and $S : \mathcal{H}_2 \to \mathcal{H}_2$ are general operators. Our study will be concerned with S and T belonging to the general class of quasi-nonexpansive operators ($\mathcal{E}_Q$), which includes commonly used classes such as firmly nonexpansive ($\mathcal{E}_{FN}$), nonexpansive ($\mathcal{E}_N$) and firmly quasi-nonexpansive (or directed, $\mathcal{E}_{FQ}$) operators; see Definition 1.1. Note that the class $\mathcal{E}_Q$ appears naturally when using subgradient projection operator techniques in solving convexly constrained problems (Yamada & Ogura, 2004; Yamada, Ogura, & Shirakawa, 2002; Yang & Zhao, 2006).

The above formalism provides an unified framework for the study of many significant real-world problems. It is worth underlining that (1.1) can be regarded as a generalization of the so-called *split feasibility problem* (introduced in Censor & Elfving (1994)):

$$\text{find } x^* \in Q_1 \text{ such that } Ax^* \in Q_2, \tag{1.2}$$

where Q_1 and Q_2 are convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Such models were successfully developed for instance in radiation therapy treatment planning, sensor networks, resolution enhancement and so on (Byrne, 2002; Censor, Bortfeld, Martin, & Trofimov, 2006; Censor, Chen, Combettes, Davidi, & Herman, 2011; Xu, 2010; Yang, 2004). Note that (1.2) is nothing but the special instance of (1.1) when taking $T = P_{Q_1}$ and $S = P_{Q_2}$ (the metric projections onto Q_1 and Q_2, respectively); see, e.g., Takahashi (2000) for more details on the metric projection. It can be also noticed that (1.1) encompasses other recently discussed inverse problems such as *split variational inequalities* (Censor, Gibali, & Reich, 2011) and *split monotone inclusions* (Moudafi, 2011).

For convenience of the reader, we recall the definitions of quasi-nonexpansive operators and other classes of operators often encountered in fixed point theory.

Definition 1.1. Consider an operator $U : \mathcal{H} \to \mathcal{H}$:

(d1) U belongs to $\mathcal{E}_{FN}$, the set of firmly nonexpansive mappings, if

$$\forall x, y \in \mathcal{H}, \quad |Ux - Uy|^2 \leqslant |x-y|^2 - |(x-y) - (Ux - Uy)|^2;$$

(d2) U belongs to $\mathcal{E}_N$, the set of nonexpansive mappings, if

$$\forall (x, y) \in \mathcal{H} \times \mathcal{H}, \quad |Ux - Uy| \leqslant |x - y|;$$

(d3) U belongs to $\mathcal{E}_{FQ}$, the set of firmly quasi-nonexpansive (or directed) mappings, if

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$$\forall (x,q) \in \mathcal{H} \times \mathrm{Fix}(U), \quad |Ux - q|^2 \leqslant |x-q|^2 - |x - Ux|^2;$$

(d4) U belongs to $\mathcal{E}_Q$, the set of quasi-nonexpansive mappings, if

$$\forall (x,q) \in \mathcal{H} \times \mathrm{Fix}(U), \quad |Ux - q| \leqslant |x - q|.$$

Remark 1.1. Clearly, we observe that $\mathcal{E}_{FN} \subset \mathcal{E}_N \subset \mathcal{E}_Q$ and that $\mathcal{E}_{FN} \subset \mathcal{E}_{FQ} \subset \mathcal{E}_Q$. It is also well-known that $\mathcal{E}_{FN}$ includes resolvents and projection operators, while $\mathcal{E}_{FQ}$ contains subgradient projection operators (Bauschke & Combettes, 2001; Byrne, 2004; Yang & Zhao, 2006).

Let us recall that (1.1) was investigated for directed operators through the following algorithm (Censor & Segal, 2009):

$$x_{n+1} = T \circ (I - \mu A^*(I - S)A)x_n, \tag{1.3}$$

where μ is some positive value and A^* is the adjoint operator of A. Clearly this iteration has been inspired from the "CQ algorithm" (Byrne, 2002, 2004), which is nothing but the special instance of (1.3) when $T = P_{Q_1}$ and $S = P_{Q_2}$ are metric projections. This latter algorithm was aimed at solving the split feasibility problem (1.2). The convergence of (1.3) and that of the CQ algorithm were first established in the finite dimensional setting, under the condition that the step-size μ satisfies

$$\mu \in \left(0, \frac{2}{|A^*A|}\right), \quad \text{where } |A^*A| \text{ is the spectral radius of the operator } A^*A. \tag{1.4}$$

Later on, further algorithmic solutions to (1.1), in general Hilbert spaces, were investigated for quasi-nonexpansive operators (even for more general demicontractive operators) through Mann-type variants of (1.3) (Moudafi, 2010, 2011). Only weak convergence results are established under a similar condition to (1.4) together with some classical demi-closedness property regarding the involved operators.

Definition 1.2. Given an operator $U : \mathcal{H} \to \mathcal{H}$, we say that $I - U$ is demiclosed (Goebel & Kirk, 1990) if

$$(z_k) \subset \mathcal{H}, \quad z_k \rightharpoonup z \quad \text{weakly}, \quad (I - U)(z_k) \to 0 \text{ strongly} \Rightarrow z \in \mathrm{Fix}(U).$$

Remark 1.2. The demiclosedness property is well-known to be satisfied for instance by certain continuous operators such as nonexpansive and more general strictly pseudocontractive ones (Browder & Petryshyn, 1967; Marino & Xu, 2007), but also by (possibly discontinuous) operators, not nonexpansive, but quasi-nonexpansive (Maingé, 2008, Lemma 4.6).

Another approach to (1.1) was proposed in Byrne, Censor, Gibali, and Reich (2011) with regard to firmly nonexpansive operators S and T through the Halpern-type variant of algorithm (1.3) given by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(T \circ (I - \mu A^*(I - S)A))(x_n), \tag{1.5}$$

where (α_n) is a slowly vanishing sequence, that is $(\alpha_n) \subset (0,1]$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n\geqslant 0} \alpha_n = +\infty$. The strong convergence of the iterations given by (1.5) was also established under the additional requirement (1.4).

To solve (1.1) relative to quasi-nonexpansive mappings S and T, Zhao and He have recently exploited a strategy similar to that used in Maingé (2010) (for the fixed point problem), by introducing relaxation parameters in their algorithm. More precisely, they proposed and studied the following "viscosity-like" iteration (see Zhao & He, 2012):

$$x_{n+1} = \alpha_n C x_n + (1 - \alpha_n) U_{w_n}(x_n), \tag{1.6}$$

where U_{w_n} corresponds to a relaxed form of the operator $U = T \circ (I + \mu A^*(S - I)A)$ (namely $U_{w_n} = (1 - w_n)I + w_n U$), μ and (w_n) being positive real numbers, $C : \mathcal{H}_1 \to \mathcal{H}_1$ is a strict contraction and (α_n) is a slowly vanishing sequence. The strong convergence of (1.6) was obtained under the additional conditions on the parameters:

$$\mu \in \left(0, \frac{1}{|A^*A|}\right); \quad 0 < \liminf_{n\to+\infty} w_n \leqslant \limsup_{n\to+\infty} w_n < 1/2. \tag{1.7}$$

Their results extend (to the split fixed point problem) that of Maingé (2010) regarding the fixed point problem, namely the special case when $\mathcal{H}_1 = \mathcal{H}_2$ and $S = I$ (so that U reduces to T).

It turns out for convergence that all the above-mentioned fixed-point methods are based upon the knowledge of the spectral radius of the operator A^*A. It is our purpose here to propose an alternative approach to solving the common fixed point problem (1.1). More precisely, we prove a strong convergent result regarding a variant of (1.6), under very classical conditions, together with a range of variable step-sizes that does not depend on $|A^*A|$. An important particular case of our less restrictive strategy is also considered, leading to Polyak-type algorithms (Polyak, 1987). Numerical experiments are also performed with regard to an inverse heat equation.

Remark 1.3. Let us emphasize that the techniques of analysis developed in this paper are different (regarding the asymptotic convergence part) from that of Zhao and He (2012). In particular, we implicitly extend the result of (at least) Maingé (2010) to the following wider and natural range of relaxation parameter

$$0 < \liminf_{n\to+\infty} w_n \leqslant \limsup_{n\to+\infty} w_n < 1. \tag{1.8}$$

Our techniques of analysis can be also applied to algorithm (1.6) so as to extend its related strong convergence result to the range of parameter (1.8). However this is out of the scope of this paper and we pay attention to a slightly different method.

2. Framework and proposed method

2.1. The involved operators

Let us recall (see Definition 1.1) that the class $\mathcal{E}_Q$ of quasi-nonexpansive mappings includes the class $\mathcal{E}_{FQ}$ of firmly quasi-nonexpansive mappings. However, as can be noticed from the literature (related to (1.1)), the conditions on the parameters for convergence of algorithms such as (1.6) are somewhat different, when dealing either with element of $\mathcal{E}_Q$ or with element of $\mathcal{E}_{FQ}$. So, in order to state precise results regarding these two situations, we will use the concept of demicontractive mappings (see, e.g., Hicks & Kubicek, 1977; Maingé & Maruster, 2011; Maruster, 1997; Moudafi, 2010).

Definition 2.1. An operator $U : \mathcal{H} \to \mathcal{H}$ is called η-demicontractive (for some real value $\eta \in (-\infty, 1)$) if, for any $(x,q) \in \mathcal{H} \times \mathrm{Fix}(U)$, it satisfies

(d1) $|Ux - q|^2 \leqslant |x - q|^2 + \eta |x - Ux|^2$, or equivalently,
(d2) $\langle x - Ux, x - q \rangle \geqslant (1/2)(1 - \eta)|x - Ux|^2$.

Remark 2.1. The equivalence between the two inequalities in Definition 2.1 can be seen from the following classical equality:

$$\forall (u,v) \in \mathcal{H}^2, \quad \langle u, v \rangle = -(1/2)|u - v|^2 + (1/2)|u|^2 + (1/2)|v|^2. \tag{2.1}$$