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Interfaces with Other Disciplines

The minimum sum representation as an index of voting power

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ABSTRACT

We propose a new power index based on the minimum sum representation (MSR) of a weighted voting game. The MSR offers a redesign of a voting game, such that voting power as measured by the MSR index becomes proportional to voting weight. The MSR index is a coherent measure of power that is ordinally equivalent to the Banzhaf, Shapley–Shubik and Johnston indices. We provide a characterization for a bicameral meet as a weighted game or a complete game, and show that the MSR index is immune to the bicameral meet paradox. We discuss the computation of the MSR index using a linear integer program and the inverse MSR problem of designing a weighted voting game with a given distribution of power.

1. Introduction

Consider a simple majority voting game, in which two voters have 49 votes each, whereas a third voter has 2 votes only. Let the weights of the voters represent their contributions to a common cause, or ownership stakes in a joint equity arrangement. Examples of the latter type include shareholder voting in corporations and country member voting in the multilateral institutions of the Bretton Woods Accord. Simple majority rule stipulates that, in order for a coalition to win, it must command at least 51 votes.

The voters have glossily unequal weights and yet are equally powerful because a coalition of any two of them wins. Why would the larger voters contribute too much, relative to the power they receive? Assuming integer weights and holding the voting weight of the smallest voter fixed at 2, the larger voters would have an incentive to reduce their voting weights to 2. This is the minimal contribution that preserves an equal distribution of power after lowering the quota to 4. The sum of voting weights, or the joint stock, shrinks from 100 to 6. In fact, we may expect a race to the bottom until the weights fall to 1 (quota 2, joint stock 3).[2]

We started from the game $[51;49,49,2]$ and ended with the game $[2;1,1,1]$. This is the minimum sum representation (MSR) of a game with three equally powerful voters. It turns out that the share in the sum of voting weights of an MSR is a valid measure of power. We call this new measure the 'MSR index'. In the above games, the power vector reads $(1/3,1/3,1/3)$. Powers according to our new measure are thus proportional to voting weights in the MSR.

We show that the MSR index is a coherent measure of power. According to Freixas and Gambarelli (1997), a measure is coherent if it: vanishes for powerless players, is invariant under isomorphisms, leads to the measured power being shared among the voters, and is strictly monotonic. The monotonicity is based on Isbell (1956) desirability relationship (see also, Taylor and Zwicker, 1999). In weighted voting games, strict Isbell monotonicity implies strict monotonicity of power in voting weight. But Isbell monotonicity also applies to more general types of complete simple games for which the desirability relationship is total. Weighed voting games are a class of complete simple games. For weighted voting games, the Freixas and Gambarelli coherency criteria are equivalent to the 'minimal adequacy postulate' by Felsenthal and Machover (1998, p. 222), plus the dominance criterion (*Ibid.*, ch. 7.6), which they find sufficiently important to disqualify two existing power indices that violate it (Deegan and Packel, 1978; Holler, 1982).

It is important to emphasize that the domain of application of the MSR index is confined to weighted voting games. In this respect the MSR index is less general than the existing power indices that can be computed for any simple voting game. It is, however, weighted voting games that are relevant to the applied power measurement and institutional design. The MSR index is ordinally equivalent to the Banzhaf, Shapley–Shubik and Johnston indices, so that all four indices produce the same power ranking in any weighted voting game.

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[2] In a non-cooperative meta-game, in which players' strategies are their voting weights and their payoff is voting power, this is unlikely to be a stable equilibrium because each player would benefit from unilaterally increasing her voting weight.

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Further exploring the properties of the MSR index, we show that it is immune to the bicameral meet paradox. A bicameral meet is a union of two voting games. Respecting bicameral meet requires that the merging of two voting games not change the relative powers of voters, who were in the same game prior to the merge. The quirk is that a bicameral meet of two complete games may not be complete, and a bicameral meet of two weighted games may not be weighted. We provide a characterization of simple voting games, in which these two properties carry over from the constituent games to the union game.

The paper is structured as follows. Sections 2 and 3 recapitulate the theoretical foundations of simple voting games and power indices, respectively. Sections 4 and 5 discuss the MSR, formulate the MSR index and establish its coherency as a measure of power. The index is uniquely determined and can be computed using an integer linear program. The inverse MSR problem of designing a weighted voting game with a given distribution of power can be solved with the same method used to compute the index. This stands in contrast to the existing power indices, whose inverse problems are significantly more difficult than direct computations. Section 6 illustrates the MSR index on two constituencies of the IMF and the German Bundestag after the general election of 2009. Vulnerabilities to certain anomalies, commonly referred to as voting paradoxes, are discussed in Section 7. In this section, we obtain the characterization for bicameral meet games and use it to show that the MSR index respects the bicameral meet postulate. In a symmetric weighted voting game, the power of a voting bloc according to the MSR index equals the sum of individual powers of its members. The MSR index is thus neutral with respect to aggregating powers in symmetric weighted voting games. This is different from the Banzhaf and Shapley–Shubik indices, which can assign more or less than the sum of individual powers to the bloc. Since a symmetric weighted voting game with a voting bloc is a particular case of a game with an a priori union, we provide a definition of the MSR index for games with a priori unions and discuss the computation of the MSR index in such games. The last section offers concluding remarks.

2. Simple voting games

A *simple voting game* (SVG) is a collection $\mathcal{W}$ of sets contained in the finite set $N = \{1, 2, \ldots, n\}$, satisfying the following properties:

(i) $N \in \mathcal{W}$;
(ii) $\emptyset \notin \mathcal{W}$;
(iii) Whenever $S \subseteq T \subseteq N$ and $S \in \mathcal{W}$, then also $T \in \mathcal{W}$.

We shall refer to N as the *assembly* of $\mathcal{W}$. The members of N are the *voters* in $\mathcal{W}$. A set of voters, a subset of N, is called a *coalition*. The *cardinal* of a set of voters S, or the number of voters in coalition S, is denoted by $|S|$.

Any member of $\mathcal{W}$ is a *winning* coalition. If $S \subseteq N$ but $S \notin \mathcal{W}$, then S is a *losing* coalition. A winning coalition is *minimal* if it has no proper winning subset. The set of winning coalitions $\mathcal{W}$, or the set of minimal winning coalitions $\mathcal{W}^m$, completely characterizes the SVG.

A voter i is a *vetoer* if $i \in S$ for all $S \in \mathcal{W}^m$. A voter i is *null* if $i \notin S$ for every $S \in \mathcal{W}^m$. A vetoer i is a *dictator* if $\mathcal{W}^m = \{\{i\}\}$. In this case, all players in $N\backslash\{i\}$ are null. A voter i in an SVG is called *trivial* if she is either a vetoer, or a null voter. A simple voting game comprising trivial voters only is called a *unanimity of a coalition game*, and has a singleton $\mathcal{W}^m = \{T\}$ for some $\emptyset \subsetneq T \subseteq N$ as the set of minimal winning coalitions.

An SVG is a *weighted voting game* (WSVG) if one can assign to each $i \in N$ a nonnegative real number w_i, and fix a real positive number q, such that

$$\mathcal{W} = \{S \subseteq N : \ w(S) \geqslant q\}, \text{ where } w(S) = \sum_{i \in S} w_i.$$

Here, w_i is the *voting weight* of voter i, and q is the number of affirmative votes required for a decision to be passed. A representation of a WSVG with a *quota* q and weights w_i for every $i \in N$ is denoted by $[q; w_1, w_2, \ldots, w_n]$, where $n = |N|$ is the number of voters. Should the vector of weights $w = (w_1, w_2, \ldots, w_n)$ be specified, we may use the shorter notation $[q; w]$ instead.

Two distinct representations of a WSVG are equivalent if they induce the same set $\mathcal{W}$. For example, $[51; 49, 49, 2] \equiv [102; 98, 98, 4]$, since $\mathcal{W} = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ in both games. This shows that the number of WSVG equivalent to a given WSVG is infinite, as rescaling the weights and quota by the same factor preserves the set of winning coalitions.

3. Coherent power measures

A *power index* is a mapping K that assigns to each SVG a vector in $\mathbb{R}^n_+$. A power index on a subclass of SVGs, say $\mathcal{S}$, is a mapping K that assigns to each game in $\mathcal{S}$ a vector in $\mathbb{R}^n_+$. For the purposes of this paper, we consider power indices on the class of WSVGs only.

In addition to nonnegativity, Freixas and Gambarelli (1997) state the additional properties any reasonable measure of power must fulfill as follows:

(i) *Null voter*: If i is a null voter in $(N, \mathcal{W})$, then $K_i(\mathcal{W}) = 0$;
(ii) *Efficiency*: $\sum_{i \in N} K_i(\mathcal{W}) = 1$;
(iii) *Invariance*: $K_i(\mathcal{W}) = K_{\pi(i)}(\mathcal{W})$ for every bijective map (isomorphism) $\pi : N \to N$, such that $S \in \mathcal{W} \iff \pi(S) \in \mathcal{W}$;
(iv) *Strong monotonicity*: if $i \succ_D j$ in $(N, \mathcal{W})$ then $K_i(\mathcal{W}) > K_j(\mathcal{W})$.

The null voter property requires the index to vanish for powerless voters. Efficiency requires voting powers to sum to unity. This normalization is appropriate when power justifies a claim on a prize to be shared among the voters (P-power in Felsenthal and Machover (2004)). The more powerful the voter, the larger the share she receives. Null voter and efficiency together imply that a dictator receives the entire prize. Invariance says that any transformation that preserves the set of winning coalitions must also preserve the distribution of power. A rescaling of quota and weights in a weighted voting game should leave the distribution of power unchanged.

Monotonicity is formulated in terms of Isbell's desirability relation. The notation $\succsim_D$ denotes a relation on N, such that $i \succsim_D j$, if $S \cup \{j\} \in \mathcal{W}$ implies $S \cup \{i\} \in \mathcal{W}$ for every $S \subseteq N\backslash\{i, j\}$. Roughly speaking, adding voter i instead of voter j to any coalition S will have the same or better effect on its decisiveness, making i a more desirable addition for the voters comprising S.

The game $(N, \mathcal{W})$ is called *complete* (CSVG) if $\succsim_D$ is a total (weak) order. Then:

$i \succ_D j$ if $i \succsim_D j$ but not $j \succsim_D i$;
$i \approx_D j$ if $i \succsim_D j$ and $j \succsim_D i$.

All WSVGs are CSVGs because $w_i \geqslant w_j$ implies $i \succsim_D j$. The class of complete simple games is thus more general than the class of weighed voting games. For $n \geqslant 6$ there exist complete SVGs that are not WSVGs and for $n \geqslant 4$ there exist SVGs that are not CSVGs.

Taylor and Pacelli (2008) offer a test of completeness. A simple game is complete if it is *swap robust*, or if a one-for-one exchange of players between any two winning coalitions S and T leaves at least one of the two coalitions winning. One of the players in the swap must belong to S but not T, and the other must belong to T but not S.

For weighted voting games, the above coherency criteria are equivalent to 'minimal adequacy postulates' for WSVGs by Felsenthal and Machover (1998), plus the strong dominance