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Decision Support

Static search games played over graphs and general metric spaces ☆

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ABSTRACT

We define a general game which forms a basis for modelling situations of static search and concealment over regions with spatial structure. The game involves two players, the searching player and the concealing player, and is played over a metric space. Each player simultaneously chooses to deploy at a point in the space; the searching player receiving a payoff of 1 if his opponent lies within a predetermined radius r of his position, the concealing player receiving a payoff of 1 otherwise. The concepts of dominance and equivalence of strategies are examined in the context of this game, before focusing on the more specific case of the game played over a graph. Methods are presented to simplify the analysis of such games, both by means of the iterated elimination of dominated strategies and through consideration of automorphisms of the graph. Lower and upper bounds on the value of the game are presented and optimal mixed strategies are calculated for games played over a particular family of graphs.

1. Introduction

In this paper, we define a general search and concealment game that takes full account of the spatial structure of the set over which it is played. The game is static in the sense that players do not move, but deploy simultaneously at particular spatial points and receive payoffs based on their relative positions. In this way, the static spatial search game (SSSG) provides a theoretical foundation for the study of the relative strategic value of different positions in a geography. Using the theory of metric spaces, we model situations in which the searching player may simultaneously search multiple locations based on concepts of distance or adjacency relative to the point at which they are deployed.

While the SSSG does build upon previous work, particularly that of Ruckle (1983) and White (1994), its simplicity and generality together with its explicit consideration of spatial structure set it apart from much of the literature (see Section 3 for a detailed review of related work) and lend it the versatility to describe games over a huge variety of different spaces. The primary contributions of this article are therefore to both propose a highly general model of spatial search and concealment situations, which unites several other games presented in the literature (see Section 4.2), and to present new propositions and approaches for the strategic analysis of such scenarios.

While this paper is theoretical in nature, the SSSG provides a framework for the analysis of a diverse range of operational research questions. Aside from explicit search and concealment scenarios, the game may be used to model situations in which some structure or region must be protected against ‘attacks’ that could arise at any spatial point; for example, the deployment of security personnel to protect cities against terrorist attacks or outbreaks of rioting, security software scanning computer networks to eliminate threats, the defence of shipping lanes against piracy, the protection of a rail network against cable theft or the deployment of stewards at public events to respond to emergency situations.

We provide a brief overview of all necessary game theoretic concepts in Section 2 and a review of the literature on games of search and security in Section 3, before formally defining the SSSG, examining its relationship to other games in the literature and presenting some initial propositions in Section 4. In Section 5, we examine the SSSG on a graph and identify upper and lower bounds on the value of such games before presenting an algorithm in Section 6 which simplifies graph games by means of the iterated elimination of dominated strategies, focusing particularly on the application of the algorithm to games played on trees. Section 7 contains further results, including a way to simplify graph games through consideration of graph automorphisms and an examination of a particular type of strategy for such games, which we describe as an “equal oddments strategy”. In Section 8, we use the concept of an equal oddments strategy to find analytic solutions for a particular family of graph games, while Section 9 forms a conclusion to the paper, containing a summary of our key results and suggestions of potential avenues for further research. Two proofs, which were too complicated to include in the main text, are presented as appendices.

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2. Game theoretic concepts

The definitions and notation relating to game theory used in this section are adapted from Blackwell and Girshick (1954) and Morris (1994).

When discussing two-player games, we assume the following definition:

Definition 2.1. Two-player games in normal form

A game in normal form between Players A and B, consists of:

- **strategy sets** Σ_A, Σ_B
- **payoff functions** $p_A,\ p_B$, with:

$$p_A : \Sigma_A \times \Sigma_B \to \mathbb{R}$$
$$p_B : \Sigma_A \times \Sigma_B \to \mathbb{R}$$

If the payoffs are such that for some constant c:

$$p_A(x,y) + p_B(x,y) = c, \quad \forall x \in \Sigma_A, \quad \forall y \in \Sigma_B$$

then the game is described as a constant-sum game.

The game is played by Players A and B simultaneously choosing **strategies** (described as **pure strategies** in cases where there may be any ambiguity) from their respective strategy sets $x \in \Sigma_A, y \in \Sigma_B$ and receiving payoffs $p_A(x,y), p_B(x,y)$. The objective of each player is to maximise their payoff.

In certain circumstances, we may allow players to adopt **mixed strategies**, whereby they choose their pure strategy according to a specified probability distribution. If Σ_A and Σ_B are finite, with:

$$\Sigma_A = \{x_1, \ldots, x_{\kappa_A}\}$$
$$\Sigma_B = \{y_1, \ldots, y_{\kappa_B}\}$$

for some positive integers κ_A, κ_B, then the mixed strategies $\boldsymbol{\sigma}_A$, $\boldsymbol{\sigma}_B$ can simultaneously be regarded as vectors:

$$\boldsymbol{\sigma}_A = (\sigma_A[x_1], \ldots, \sigma_A[x_{\kappa_A}]) \in [0,1]^{\kappa_A}$$
$$\boldsymbol{\sigma}_B = (\sigma_B[y_1], \ldots, \sigma_B[y_{\kappa_B}]) \in [0,1]^{\kappa_B}$$

and as functions, which allocate probabilities to pure strategies:

$$\sigma_A : \Sigma_A \to [0,1]$$
$$x \mapsto \sigma_A[x]$$
$$\sigma_B : \Sigma_B \to [0,1]$$
$$y \mapsto \sigma_B[y]$$

$$\sum_{x \in \Sigma_A} \sigma_A[x] = \sum_{y \in \Sigma_B} \sigma_B[y] = 1$$

The following definitions relate to the maximum expected payoff that players can guarantee themselves through careful choice of their mixed strategies:

Definition 2.2. Values of the game

Given a two-player game, the values of the game u_A, u_B to Players A and B respectively, are defined as:

- $u_A = \max_{\boldsymbol{\tau}_A} \min_{\boldsymbol{\tau}_B} E[p_A(\boldsymbol{\tau}_A, \boldsymbol{\tau}_B)]$
- $u_B = \max_{\boldsymbol{\tau}_B} \min_{\boldsymbol{\tau}_A} E[p_B(\boldsymbol{\tau}_A, \boldsymbol{\tau}_B)]$

where $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$ range across all possible mixed strategies for Players A and B respectively and

$$E[p_A(\boldsymbol{\tau}_A, \boldsymbol{\tau}_B)]$$
$$E[p_B(\boldsymbol{\tau}_A, \boldsymbol{\tau}_B)]$$

represent the expected payoffs to each player, given that they respectively adopt mixed (or pure) strategies $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$.

Definition 2.3. Optimal mixed strategies

Given a two-player constant-sum game, where the payoffs sum to $c \in \mathbb{R}$, mixed strategies $\boldsymbol{\sigma}_A$, $\boldsymbol{\sigma}_B$ for Players A and B are described as **optimal** if and only if:

- $\min_{\boldsymbol{\tau}_B} E[p_A(\boldsymbol{\sigma}_A, \boldsymbol{\tau}_B)] = u_A$
- $\min_{\boldsymbol{\tau}_A} E[p_B(\boldsymbol{\tau}_A, \boldsymbol{\sigma}_B)] = u_B$

where $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$ range across all possible mixed strategies for Players A and B respectively.

For a constant-sum game, where the payoffs sum to $c \in \mathbb{R}$, we have:

$$u_A + u_B = c \qquad (1)$$

Also, provided that Σ_A and Σ_B are finite, optimal mixed strategies are guaranteed to exist for both players.

Both of these facts are consequences of the Minimax Theorem (see Morris, 1994, p. 102).

Given a constant-sum two-player game with finite strategy sets, a **solution** of the game comprises optimal mixed strategies $\Sigma_A,\ \Sigma_B$ and values $u_A,\ u_B$ for each Player.

The following definition allows for a crude comparison of the efficacy of different strategies.

Definition 2.4. Strategic dominance and equivalence

Consider a two-player game with strategy sets $\Sigma_A,\ \Sigma_B$ and payoff functions $p_A,\ p_B$. Given particular pure strategies $x_1,\ x_2 \in \Sigma_A$ for Player A, we have:

- x_2 **very weakly dominates** x_1 if and only if:
$$p_A(x_2, y) \geqslant p_A(x_1, y), \quad \forall y \in \Sigma_B$$
- x_2 **weakly dominates** x_1 if and only if:
$$p_A(x_2, y) \geqslant p_A(x_1, y), \quad \forall y \in \Sigma_B$$

and $\exists y^* \in \Sigma_B$ such that:
$$p_A(x_2, y^*) > p_A(x_1, y^*)$$
- x_2 **strictly dominates** x_1 if and only if:
$$p_A(x_2, y) > p_A(x_1, y), \quad \forall y \in \Sigma_B$$
- x_2 is **equivalent** to x_1 if and only if:
$$p_A(x_2, y) = p_A(x_1, y), \quad \forall y \in \Sigma_B$$

Since the designation of the players as A and B is arbitrary, obtaining corresponding definitions of strategic dominance and equivalence for Player B is simply a matter of relabelling.

Note that weak dominance, strict dominance and equivalence are all special cases of very weak dominance. Also, strict dominance is a special case of weak dominance.

In this paper, weak dominance is of most relevance. Therefore, for reasons of clarity, the terms "dominance" and "dominated strategies" will be used to refer to weak dominance unless otherwise stated.

Since a player aims to maximise his or her payoff, we would intuitively expect that they should not play any dominated strategies.

For a general definition of dominance in game theory, see Leyton-Brown and Shoham (2008, pp. 20–23), from which the above definition was adapted.

3. Games of search and security: a review

Games of search and concealment, in which one player attempts to hide themselves or to conceal some substance in a