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On allocation of redundant components for systems with dependent components

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In this paper we consider the problem of optimal allocation of a redundant component for series, parallel and k-out-of-n systems of more than two components, when all the components are dependent. We show that for this problem is naturally to consider multivariate extensions of the joint bivariates stochastic orders. However, these extensions have not been defined or explicitly studied in the literature, except the joint likelihood ratio order, which was introduced by Shanthikumar and Yao (1991). Therefore we provide first multivariate extensions of the joint stochastic, hazard rate, reversed hazard rate order and next we provide sufficient conditions based on these multivariate extensions to select which component performs the redundancy.

1. Introduction

Allocation of redundant component in a system in order to optimize, in some sense, the lifetime of the system is an important problem in reliability, from an applied and theoretical point of view. In the literature we can found a great number dealing with this subject. In Boland and Proschan (1994) we can find several results and references. More recent results can be found in the papers by Mi (1999), Valdés and Zequeira (2003), Romera et al. (2004), Yalaoui et al. (2005), Valdés and Zequeira (2006), Ha and Kuo (2006), Li and Hu (2008), Hu and Wang (2009), Misra et al. (2009), Li and Ding (2010), Valdés et al. (2010), Ding and Li (2012) and Zhao et al. (2012), among others. Whereas this problem has been extensively treated for the case of independent components, the case of dependent components has not received too much attention. Only the papers by Kotz et al. (2003), da Costa Bueno (2005) and da Costa Bueno and Martins do Carmo (2007) deals with this problem. Some of these results can be found also in Section 10.3 in Lai and Xie (2006). Recently Belzunce et al. (2011) have considered the problem of allocation of redundant components for series and parallel system in the case of two dependent components. In this paper two commonly used redundancies are discussed: active redundancy, which stochastically leads to consideration of the maximum of random variables, and standby redundancy, which stochastically leads to consideration of the convolution of random variables.

The purpose of this paper is to extend the results given by Belzunce et al. (2011) to the case of systems of more than two components, when all the components are dependent. Let us consider a system with n components with random lifetimes $T_1, T_2, \ldots, T_n$ and an additional component, with random lifetime S, that can be put in active or standby redundancy with anyone of the n components. Then, we can consider n possibly different systems, depending on which component we perform the redundancy, and the problem is which one of the n system has a larger lifetime in some probabilistic sense. We initially consider series and parallel systems, and later we consider the problem of allocating a redundant spare in a k-out-of-n system of n components. A k-out of-n system is a system of n components which functions if, and only if, at least k of its components function and whose lifetime we will be denote by $\tau_k\{T_1, \ldots, T_n\}$. Series and parallel systems are particular cases of k-out-of-n and more specifically, we have that $\tau_n\{T_1, \ldots, T_n\}$ and $\tau_1\{T_1, \ldots, T_n\}$ are the lifetimes of the resulting series system and parallel system, respectively. For k-out of-n system, one may refer to Barlow and Proschan (1981) for a comprehensive discussion.

For a treatment of this problem, previously it has been necessary to consider multivariate extensions of the joint bivariate orders introduced by Shanthikumar and Yao (1991) and Shanthikumar et al. (1991). For the moment we have been able only to applied the joint stochastic orders for the case of one redundant component. It remains as an open problem the case of $s > 1$ redundant components, like in Mi (1999), Hu and Wang (2009) or Ding and Li (2012).

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The structure of this paper is as follows. In Section 2 we consider several notions of stochastic orders, we study multivariate extensions of the joint stochastic, hazard rate and reversed hazard orders. We establish also some results which will be used in the proofs of Section 3. The main results are presented in Section 3 and they will be compared to the existing ones in the case of independence. We conclude, in Section 4, with examples where the results can be applied.

In this paper, for any random variable X and an event A, we denote by $\{X|A\}$ any random variable whose distribution is the conditional distribution of X given A. The random variables considered in Section 3 are assumed to be continuous and nonnegative. Given an n-dimensional random vector $(X_1,\ldots,X_n)$ with joint distribution function F, we denote by $\overline{F}(x_1,\ldots,x_n) \equiv P(X_1 > x_1,\ldots,X_n > x_n)$ its joint survival function.

2. Stochastic orders

In order to select, among two systems, which one has a "larger" random lifetime, one of the most important tools is the comparison of random lifetimes in terms of several notions of stochastic orders. Several criteria have been defined to compare the "magnitude" of two random variables, we recall first the definition of some of these stochastic orders (see Müller and Stoyan, 2002; Shaked and Shanthikumar, 2007 for definitions, properties and references).

Definition 2.1. Let X and Y be two random variables, we say that X is smaller than Y in the usual stochastic order, denoted by $X \leqslant_{st} Y$, if $\overline{F}(t) = P[X > t] \leqslant P[Y > t] = \overline{G}(t),$ for all $t \in \mathbb{R}$, or equivalently, if $E[\phi(X)] \leqslant E[\phi(Y)]$ for all increasing function ϕ for which previous expectations exist.

The definition of the stochastic order is a way to formalize the idea that the random variable X is less likely than Y to take on large values. For the case of two random vectors we have the following extension.

Definition 2.2. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors, we say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the usual stochastic order, denoted by $\boldsymbol{X} \leqslant_{\text{st}} \boldsymbol{Y}$, if $E[\phi(\boldsymbol{X})] \leqslant E[\phi(\boldsymbol{Y})]$ for all increasing function $\phi : \mathbb{R}^n \to \mathbb{R}$, for which previous expectations exist.

Two properties that will be used along the paper are the preservation under increasing transformations and under mixtures.

Lemma 2.1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors.*

(*a*) *If $\boldsymbol{X} \leqslant_{st} \boldsymbol{Y}$ then $\phi(\boldsymbol{X}) \leqslant_{st} \phi(\boldsymbol{Y})$, for any increasing function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^k$.*
(*b*) *Let $\boldsymbol{\Theta}$ be an m-dimensional random vector, if $[\boldsymbol{X}|\boldsymbol{\Theta} = \boldsymbol{\theta}] \leqslant_{st} [\boldsymbol{Y}|\boldsymbol{\Theta} = \boldsymbol{\theta}]$, for all $\boldsymbol{\theta}$ in the support of $\boldsymbol{\Theta}$, then $\boldsymbol{X} \leqslant_{st} \boldsymbol{Y}$.*

Some other notions have been found of interest to compare random variables.

Definition 2.3. Let X and Y two random variables with distribution functions F and G, respectively, we say that X is smaller than Y in the hazard rate order, denoted by $X \leqslant_{\text{hr}} Y$, if $\overline{F}(x)\overline{G}(y) \geqslant \overline{F}(y)\overline{G}(x)$ for all $x \leqslant y$. Additionally we say that X is smaller than Y in the reversed hazard rate order, denoted by $X \leqslant_{\text{rh}} Y$, if $F(x)G(y) \geqslant F(y)G(x)$ for all $x \leqslant y$.

In the case of absolutely continuous random variables we can consider also the following criteria.

Definition 2.4. Let X and Y two absolutely continuous random variables with density functions f and g, respectively, we say that X is smaller than Y in the likelihood ratio order, denoted by $X \leqslant_{\text{lr}} Y$, if $f(x)g(y) \geqslant f(y)g(x)$ for all $x \leqslant y$.

Among these orders we have the following relationships:

$$X \leqslant_{\text{lr}} Y \Rightarrow X \leqslant_{[\text{hr,rh}]} Y \Rightarrow X \leqslant_{\text{st}} Y.$$

Another criteria to compare random lifetimes used in this context is the following.

Definition 2.5. Let X and Y two random variables, we say that X is smaller than Y in the preference order, denoted by $X \leqslant_{\text{pr}} Y$, if $P(X > Y) \leqslant P(Y > X)$.

No relationships are known among this notion and the hr, rh and lr orders, and it is well known (see Blyth, 1972) that

$$X \leqslant_{\text{st}} Y \not\Rightarrow (\not\Leftarrow) X \leqslant_{pr} Y.$$

Except for the preference order, the stochastic orders considered previously only take into account the information provided by the marginal distributions to compare the random variables. Shanthikumar and Yao (1991) and Shanthikumar et al. (1991) introduce some stochastic orders that take into account the dependence among the random variables. To give these definitions we recall first, the definition of the following sets of functions.

Definition 2.6. Let us denote by D the set of functions $D = \{g | g : \mathbb{R}^2 \mapsto \mathbb{R}\}$, we consider the following sets:

(a) $G_{\text{lr}} = \{g \in D\colon g(x, y) \leqslant g(y, x)$, for all $x \leqslant y\}$.
(b) $G_{\text{hr}} = \{g \in D\colon g(y, x) - g(x, y)$ is increasing in y, for all $x \leqslant y\}$.
(c) $G_{\text{rh}} = \{g \in D\colon g(x, y) - g(y, x)$ is increasing in x, for all $x \leqslant y\}$.
(d) $G_{\text{st}} = \{g \in D\colon g(y, x) - g(x, y)$ is increasing in y, for all $x\}$.

We recall now the following definitions.

Definition 2.7. Given two random variables X and Y, we say that X is smaller than Y in the joint usual stochastic [hazard rate, reversed hazard rate, likelihood ratio] order, denoted by $X \leqslant_{\text{st:j [hr:j, rh:j, lr:j]}} Y$, if $E[g(X, Y)] \leqslant E[g(Y, X)]$, for all $g \in G_{\text{st[hr, rh, lr]}}$.

These joint stochastic orders have been used by Belzunce et al. (2011) to provide results for the allocation of redundant components in series and parallel systems with two dependent components. The purpose of this paper is to extend these results to the general case with more than two dependent components and also to provide results in the general case of k-out-of-n systems. Therefore it is natural to consider extensions of the joint stochastic orders from the bivariate to the multivariate case. However, these extensions have not been defined or explicitly studied in the literature, except for the joint likelihood ratio order, which was introduced by Shanthikumar and Yao (1991). To give this extension, we need to recall the following definitions.

Definition 2.8. Let $\boldsymbol{x} = (x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)$ and $\boldsymbol{x}' = (x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)$ be two vectors in $\mathbb{R}^n$. We said that $\mathbf{x}'$ is a simple transposition of $\mathbf{x}$, denoted by $\mathbf{x}' \leqslant^t \mathbf{x}$, if $x_i < x_j$, with $i < j \in \{1,\ldots,n\}$.

Definition 2.9. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$. We said that $\mathbf{x} \leqslant^a \mathbf{y}$ if there exist $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r \in \mathbb{R}^n$ such that $\mathbf{x} \leqslant^t \mathbf{x}_1 \leqslant^t \mathbf{x}_2 \leqslant^t \cdots \leqslant^t \mathbf{x}_r \leqslant^t \mathbf{y}$, that is, $\mathbf{x}$ can be obtained from $\mathbf{y}$ by a finite sequence of simple transpositions.