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Discrete Optimization

Maximal and supremal tolerances in multiobjective linear programming

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ABSTRACT

Let a multiobjective linear programming problem and any efficient solution be given. Tolerance analysis aims to compute interval tolerances for (possibly all) objective function coefficients such that the efficient solution remains efficient for any perturbation of the coefficients within the computed intervals. The known methods either yield tolerances that are not the maximal possible ones, or they consider perturbations of weights of the weighted sum scalarization only. We focus directly on perturbations of the objective function coefficients, which makes the approach independent on a scalarization technique used. In this paper, we propose a method for calculating the supremal tolerance (the maximal one need not exist). The main disadvantage of the method is the exponential running time in the worst case. Nevertheless, we show that the problem of determining the maximal/supremal tolerance is NP-hard, so an efficient (polynomial time) procedure is not likely to exist. We illustrate our approach on examples and present an application in transportation problems. Since the maximal tolerance may be small, we extend the notion to individual lower and upper tolerances for each objective function coefficient. An algorithm for computing maximal individual tolerances is proposed.

1. Introduction

This paper is a contribution to postoptimal analysis in multiobjective linear programming, namely to tolerance analysis. Postoptimal analysis is a fundamental tool to study effects of various uncertainties and data perturbations on the model. The standard sensitivity analysis [25,27,28] inspects behavior of one coefficient perturbation. Contrary, tolerance analysis was developed to handle simultaneous and independent variations of several coefficients. Thus it is a more powerful technique enabling the decision maker to handle more complex perturbations.

Tolerance analysis was pioneered by Wendell [31,32] in linear programming, and then investigated by many researchers; see e.g. [1,15,30,34]. It found applications not only in mathematical programming, but for example in linear regression [17,29], too.

In multiobjective linear programming, tolerance analysis was adapted by a few of ways. Since multiobjective linear programming problems are often solved by a weighted sum scalarization, the first approach concerns tolerance analysis of the objective function weights [4,5,10,19]. In this approach, one solves a weighted sum scalarization problem by a simplex method, and then determines the maximal tolerances for the weights while retaining the optimal basis. Badra [2] extended the method to calculate a percentage tolerance allowing perturbations in both the weighted sum of the objective function coefficients and in the right hand side terms while remaining the same solution optimal.

There are several disadvantages of these approaches. First, it gives no information about admissible perturbations of the original objective function coefficients, which is a serious drawback since sensitivity analysis of the original input data is of high importance. Second, it is highly dependent on the weighted sum scalarization and on the simplex method, and cannot be easily adapted to other scalarization technique and to other linear programming solvers. Further, basis invariancy based sensitivity approach is known to be restricted and does not yield the highest possible perturbation ranges [13,15,18].

To overcome these drawbacks, another way of research was to adapt tolerance analysis directly on the objective function coefficients, and also not to rely on the simplex method based analyses. This approach was followed by Hladík [11], who proposed an algorithm for satisfactory large, but not necessarily maximal, tolerances. Note that the maximal tolerances need not exist, so we will speak more about *supremal* tolerances instead. Sitarz [26] calculates an upper bound on the supremal additive tolerance. An extension to individual tolerances was given in [12]. So far, there has been no method known for computing the supremal tolerances, only the afore-mentioned lower and upper bounds. Herein, we present such an algorithm to compute the supremal tolerances.

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The paper is organized as follows. After some preliminaries (Section 2) we propose a formula to compute the supremal tolerances in terms of an optimization problem (Section 3). In Section 3.2 we show how to practically solve the optimization problem by decomposing into orthants; a reduction method for the number of orthants to be inspected is proposed, too. A procedure to test whether the calculated supremal tolerance is also the maximal one is presented in Section 3.3. NP-hardness of determining the maximal/supremal tolerance is proved in Section 3.1. Section 3.4 is concerned with an upper bound on the supremal tolerance by using edges emerging from a given vertex. Some illustrative examples, an application in transportation problems and a limited numerical study is given in Section 3.5. In Section 4, we extend the tolerance analysis problem to finding individual lower and upper tolerances for the particular objective function coefficients, and propose a method for calculating maximal individual tolerances. Section 5 concludes and states some open problems.

2. Preliminaries and problem statement

Let us introduce some notation first. By an interval matrix we mean a family of matrices

$$[\underline{M},\overline{M}] = \{M \in \mathbb{R}^{m\times n} :\ \underline{M} \leqslant M \leqslant \overline{M}\},$$

where $\underline{M} \leqslant \overline{M}$ are given matrices, and relation $\leqslant$ is understood entrywise. The relation $x \gneqq y$ denotes in short that $x \geqslant y$ and $x \neq y$, $\mathrm{diag}(v)$ stands for the diagonal matrix with entries $v_1, \ldots, v_n$, and $\mathrm{sgn}(v)$ for the sign vector of a vector v. The ith row of a matrix A is denoted by $A_{i\cdot}$, and $e = (1, \ldots, 1)^T$ is a vector of ones.

Consider a multiobjective linear programming problem

$$\begin{aligned} \max\quad & Cx \\ \text{s.t.}\quad & Ax \leqslant b, \end{aligned} \tag{1}$$

where $C \in \mathbb{R}^{p\times n}$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. A feasible solution x^* is called *efficient* if there is no feasible x such that $Cx \gneqq Cx^*$.

Now, let $G \geqslant 0$ be an $p \times n$ matrix and consider the interval matrix $[C - \delta G, C + \delta G]$ with parameter $\delta > 0$, and x^* an efficient solution to (1). A non-negative value δ is called *admissible tolerance* if x^* remains efficient for all $C' \in [C - \delta G, C + \delta G]$. Herein, G represents perturbation scales for objective function coefficients. It is usually set up as $G_{ij} = |C_{ij}|$ for relative (percentage) tolerances, $G_{ij} = 1$ for additive tolerances, and $G_{ij} = 0$ in case when perturbation of C_{ij} is not in interest. However, they can be set up in any other way according to the decision maker preferences and importances of particular coefficients.

Our aim is to calculate the maximal admissible tolerance of an efficient solution x^*. We impose no other assumption on x^*, that is, it may be basic or non-basic, degenerate or non-degenerate, and it may be an extreme point or not. Thus, our approach is independent on the solution method used for calculating x^*.

Formally, we define and denote the maximal tolerance as follows

$$\begin{aligned} \delta^{\max} := \max\quad & \delta \\ \text{s.t.}\quad & x^* \text{is efficient } \forall C' \in [C - \delta G, C + \delta G],\quad \delta \geqslant 0. \end{aligned}$$

Note that the maximal tolerance need not exist; see Example 1. That is why we focus more on calculation of the supremal tolerance

$$\begin{aligned} \delta^{\sup} := \sup\quad & \delta \\ \text{s.t.}\quad & x^* \text{ is efficient } \forall C' \in [C - \delta G, C + \delta G],\quad \delta \geqslant 0. \end{aligned}$$

Once the supremal tolerance $\delta^{\sup}$ is computed, $\delta^{\sup} - \varepsilon$ is an admissible tolerance for arbitrarily small $\varepsilon > 0$, but $\delta^{\sup}$ itself may not be admissible.

3. Maximal and supremal tolerances

Let x^* be a feasible solution and $I(x^*) = \{i;\ A_{i\cdot}x = b_i\}$ its active set. The tangent cone at x^* is described

$$A_{i\cdot}(x - x^*) \leqslant 0, \quad i \in I(x^*).$$

For simplicity, we denote the system by $A^1(x - x^*) \leqslant 0$. It is known [6] that x^* is efficient iff there is no dominated solution within the tangent cone, that is, the system

$$A^1(x - x^*) \leqslant 0, \quad C(x - x^*) \gneqq 0, \tag{2}$$

or

$$A^1(x - x^*) \leqslant 0, \quad C(x - x^*) = y \geqslant 0,\ \ e^T y = 1 \tag{3}$$

has no solution. We utilize this characterization of efficiency to derive more general robust characterization of efficiency in the following lemma, and to state our main computational result in the sequel.

Lemma 1. *Let x^* be an efficient solution to (1). Then x^* is efficient for each $C' \in [C - \delta G, C + \delta G]$ iff the system*

$$A^1(x - x^*) \leqslant 0, \quad C(x - x^*) + \delta G|x - x^*| \gneqq 0 \tag{4}$$

has no solution.

Proof. "Sufficiency." Let $C' \in [C - \delta G, C + \delta G]$ and suppose that x^* is not efficient for C', that is, there is a solution x to

$$A^1(x - x^*) \leqslant 0, \quad C'(x - x^*) \gneqq 0.$$

Then x fulfills also

$$\begin{aligned} 0 \lneqq C'(x - x^*) &= C(x - x^*) + (C' - C)(x - x^*) \\ &\leqslant C(x - x^*) + |C' - C||x - x^*| \\ &\leqslant C(x - x^*) + \delta G|x - x^*|. \end{aligned}$$

"Necessity." Now, suppose that x solves (4). Putting

$$C' := C + \delta G \ \mathrm{diag}\ (\mathrm{sgn}(x - x^*)) \in [C - \delta G, C + \delta G],$$

we get

$$\begin{aligned} C'(x - x^*) &= C(x - x^*) + \delta G \ \mathrm{diag}\ (\mathrm{sgn}(x - x^*))(x - x^*) \\ &= C(x - x^*) + \delta G|x - x^*| \gneqq 0. \end{aligned}$$

Thus x^* is not efficient for C'. □

Theorem 1. *Let x^* be an efficient solution to (1). Then*

$$\begin{aligned} \delta^{sup} = \min\quad & \delta \\ \text{s.t.}\quad & A^1(x - x^*) \leqslant 0, \quad C(x - x^*) + \delta G|x - x^*| \geqslant 0, & \text{(5a)} \\ & e^T G|x - x^*| = 1, \quad \delta \geqslant 0. & \text{(5b)} \end{aligned}$$

Proof. By Lemma 1, x^* is efficient for each $C' \in [C - \delta G, C + \delta G]$ iff the system (4) has no solution. Thus we seek the supremal δ such that the system (4) has no solution. Instead, we compute the infimal $\delta \geqslant 0$ such that the system (4) has a solution, that is

$$\begin{aligned} \inf\quad & \delta \\ \text{s.t.}\quad & A^1(x - x^*) \leqslant 0, \quad C(x - x^*) + \delta G|x - x^*| \gneqq 0,\ \ \delta \geqslant 0. \end{aligned} \tag{6}$$

Its equivalent form is (5). The reason is the following. Let (δ, x) be a feasible solution to (6). If $e^T G|x - x^*| = 0$ then $G|x - x^*| = 0$ and $C(x - x^*) \gneqq 0$, meaning that x^* cannot be efficient. Otherwise, if $e^T G|x - x^*| > 0$ then $\left(\delta, \frac{1}{e^T G|x-x^*|}(x - x^*) + x^*\right)$ solves (5). Let (δ, x) be