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Decision Support

Cross-efficiency evaluation with directional distance functions

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ABSTRACT

This paper extends the cross-efficiency evaluation for use with directional distance functions. Cross-efficiency evaluation has been developed with oriented Data Envelopment Analysis (DEA) models, so the extension proposed here is aimed at providing a peer-evaluation of decision making units (DMUs) based on measures that account for the inefficiency both in inputs and in outputs simultaneously. We explore the duality relations regarding the models of directional distance functions and define the cross-efficiencies on the basis of the equivalences with some fractional programming problems. Finally, we address in this new context the problem with the alternate optima for the weights and propose some models that implement different alternative secondary goals.

1. Introduction

Cross-efficiency evaluation was introduced in Sexton et al. (1986) and Doyle and Green (1994) as an extension of DEA aimed at ranking the DMUs. DEA provides a self-evaluation of the DMUs by using input and output weights that are unit-specific, and this makes impossible to derive an ordering. In contrast, cross-efficiency evaluation provides a peer-appraisal of the DMUs in which each unit is also assessed with the DEA weights of the others. To be specific, this methodology uses the DEA weights of all the DMUs in the calculation of the so-called cross-efficiencies, which are the usual ratios of a weighted sum of outputs to a weighted sum of inputs obtained for each unit with such weights. The cross-efficiency scores of the different units are the average of their cross-efficiencies, and such scores can therefore be used to rank the DMUs. Cross-efficiency evaluation has been used in different context: For some recent applications see Chen (2002), to electricity distribution sector, Lu and Lo (2007), to economic–environmental performance, Wu et al. (2009), to sport at the Summer Olympic, and Falagario et al. (2012), to public procurement.

Cross-efficiency evaluation has been developed theoretically by using the Charnes, Cooper and Rhodes (CCR) DEA model (Charnes et al., 1978), either input or output oriented, for the assessment of efficiency, and the applications that have been carried out obviously use those models. Nevertheless, when assessing efficiency is concerned, it is sometimes desirable that the measures provided account for the input excesses and output shortfalls simultaneously. The hyperbolic efficiency measure (Färe et al., 1985), the directional distance function (Chambers et al., 1996, 1998), the geometric distance function (Portela and Thanassoulis, 2002) and the measures based on the slacks of additive-type models like the Enhanced Russell Graph efficiency measure (Pastor et al., 1999, and Tone, 2001) are examples of non-oriented efficiency measures that can be found in the related literature. In this paper, we deal specifically with directional distance functions. The fact that the directional distance function combines features of both input-oriented and output-oriented CCR models may lead to a more complete ranking of the DMUs than either of the oriented models. It should be noted, however, that the efficiency measures provided by the directional models might have the same difficulties as the classical DEA efficiency scores when used for ranking purposes, as they result from DMU-specific weights. For this reason, we propose here to extend the cross-efficiency evaluation for use with directional distance functions. Thus, with this approach we will be able to rank the DMUs by using measures that account for the inefficiency both in inputs and in outputs simultaneously.

Like in the standard cross-efficiency evaluation, the cross-efficiencies in this paper are defined as the ratios involved in some fractional programs that are equivalent to the dual problems to some formulations of the directional models. The cross-efficiency scores are defined as usual as the average of the cross-efficiencies. We also address here the problems with the alternate optima of the DEA weights, which is perhaps the main difficulty with the cross-efficiency evaluation. The existence of alternate optima for the weights may lead to different cross-efficiency scores, and consequently to different rankings of units, depending on the choice that each DMU makes. As a potential remedy to the possible influence of this difficulty, it has been suggested the use of alternative secondary goals to the choice of weights among the alternative optimal solutions. The well-known benevolent and aggressive formulations (Sexton et al., 1986; Doyle and Green, 1994) are examples of models that use an additional criterion for the selection of weights (see Liang et al. (2008) and Wang and Chin (2010b) for

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extensions of these models). Others would include Wang and Chin (2010a), Ramón et al. (2010, 2011), and Wang et al. (2012). In this paper, we develop models that implement benevolent and aggressive criteria to the choice of weights in the context of the cross-efficiency evaluation based on directional distance functions we present here.[1]

The paper unfolds as follows: Section 2 includes a brief description of the standard cross-efficiency evaluation in the context of the oriented DEA models. In Section 3 we develop the models that allow us to define the cross-efficiencies in the context of the measurement of efficiency with directional distance functions. In Section 4 we discuss the choice of weights among alternate optima and show how to adapt to this new framework some of the existing approaches that use alternative secondary goals to that end. Section 5 illustrates the proposed approach with a data set of the literature. Section 6 concludes.

2. DEA and cross-efficiency evaluation

Throughout the paper we assume that we have n DMUs that use m inputs to produce s outputs. These can be described by means of the vectors (X_j, Y_j), $j = 1, \ldots, n$. We also denote by X the $m \times n$ matrix of input vectors and by Y the $s \times n$ matrix of output vectors. The standard cross-efficiency evaluation uses the oriented CCR DEA models for the calculation of the cross-efficiencies. If, for example, an input orientation is assumed, the efficiency of a given DMU_0 is the optimal value of the following problem

$$\begin{aligned} \text{Max} \quad & \theta_0 = \frac{u'Y_0}{v'X_0} \\ \text{s.t.}: \quad & \frac{u'Y_j}{v'X_j} \leqslant 1 \quad j = 1, \ldots, n \\ & v \geqslant 0_m, u \geqslant 0_s \end{aligned} \tag{1}$$

This is the CCR model in its ratio form. By using the results on linear fractional programming in Charnes and Cooper (1962), (1) can be converted into the following linear problem (which is the so-called dual multiplier formulation)

$$\begin{aligned} \text{Max} \quad & u'Y_0 \\ \text{s.t.}: \quad & v'X_0 = 1 \\ & -v'X_j + u'Y_j \leqslant 0 \quad j = 1, \ldots, n \\ & v \geqslant 0_m, \; u \geqslant 0_s \end{aligned} \tag{2}$$

In cross-efficiency evaluations we use the optimal solutions of (2) to calculate the cross-efficiencies. To be specific, if $(v_1^d, \ldots, v_m^d, u_1^d, \ldots, u_s^d)$ is an optimal solution of (2) for a given DMU_d, then the cross-efficiency of DMU_j, $j = 1, \ldots, n$, obtained with the weights of DMU_d is the following

$$E_{dj} = \frac{u^{d\prime}Y_j}{v^{d\prime}X_j} \tag{3}$$

Then, the cross-efficiency score of DMU_j, $j = 1, \ldots, n$, is usually defined as the average of its cross-efficiencies obtained with the weights of all the DMUs. That is, the cross-efficiency score of DMU_j is defined as

$$\overline{E}_j = \frac{1}{n}\sum_{d=1}^{n} E_{dj}, \quad j = 1, \ldots, n. \tag{4}$$

The cross-efficiency score $\overline{E}_j$ provides a peer-evaluation of DMU_j, and these values can be used for ranking the DMUs.

[1] See Yang et al. (2012) and Alcaraz et al. (2013) for two different approaches that consider all the optimal DEA weights of all the DMUs in the cross-efficiency evaluation.

3. Directional distance functions and cross-efficiency evaluation

The standard cross-efficiency evaluation uses the CCR DEA model and therefore is based on efficiency measures that only account for the inefficiency either in inputs or in outputs. In this paper, we propose to extend the idea of the cross-efficiency evaluation for use with efficiency measures that are non-oriented, i.e., which account for the inefficiency both in inputs and in outputs simultaneously. To do it, we specifically propose to use directional distance functions. These provide efficiency measures that reflect the potential of a given DMU_0 for increasing outputs while reducing inputs simultaneously along a given direction determined by the vector (g^X, g^Y) (Chambers et al., 1998). If (g^X, g^Y) is chosen as $(-X_0, Y_0)$, then a value of the directional distance function, β_0, can be obtained as the optimal value of the following CRS DEA formulation

$$\begin{aligned} \text{Max} \quad & \beta_0 \\ \text{s.t.} \quad & X\lambda \leqslant (1-\beta_0)X_0 \\ & Y\lambda \geqslant (1+\beta_0)Y_0 \\ & \lambda \geqslant 0_n, \; \beta_0 \text{ free} \end{aligned} \tag{5}$$

β_0 is a measure of inefficiency, in the sense of Farrell (1957), which takes values in [0,1). DMU_0 is efficient if $\beta_0 = 0$, and the larger the value of β_0 the higher the inefficiency of DMU_0. If, for example, $\beta_0 = 0.1$, this means that DMU_0 may expand all its outputs by 10% while at the same time reducing its inputs by 10% in order to achieve the efficiency.

The developments below are intended to provide an approach to the cross-efficiency evaluation based on directional distance functions. We start with the following reformulation of (5)

$$\begin{aligned} \text{Max} \quad & \delta_0 \\ \text{s.t.} \quad & X\lambda \leqslant (2-\delta_0)X_0 \\ & Y\lambda \geqslant \delta_0 Y_0 \\ & \lambda \geqslant 0_n, \; \delta_0 \text{ free} \end{aligned} \tag{6}$$

Model (6) is actually the result of the change of variable $\delta_0 = 1 + \beta_0$ in (5). This formulation allows us to easily find a fractional program like (1) which is equivalent to the dual to (6), so we can define the cross-efficiencies in a similar manner as in the standard cross-efficiency evaluation (a problem very similar to (6) can be found in Asmild et al. (2007)). The dual problem to (6) is the following model

$$\begin{aligned} \text{Min} \quad & 2v'X_0 \\ \text{s.t.}: \quad & v'X_0 + u'Y_0 = 1 \\ & -v'X_j + u'Y_j \leqslant 0 \quad j = 1, \ldots, n \\ & v \geqslant 0_m, \; u \geqslant 0_s \end{aligned} \tag{7}$$

And (7) is the result of the conversion of the following fractional program by using again the results in Charnes and Cooper (1962)

$$\begin{aligned} \text{Min} \quad & \frac{2v'X_0}{v'X_0 + u'Y_0} \\ \text{s.t.}: \quad & \frac{2v'X_j}{v'X_j + u'Y_j} \geqslant 1 \quad j = 1, \ldots, n \\ & v \geqslant 0_m, \; u_s \geqslant 0_s \end{aligned} \tag{8}$$

Therefore, we can obtain the value β_0 by solving (8) as follows

$$\beta_0 = \delta_0 - 1 = \frac{2v'X_0}{v'X_0 + u'Y_0} - 1 = \frac{v'X_0 - u'Y_0}{v'X_0 + u'Y_0}. \tag{9}$$

To be specific, the value β_0 can be obtained as in (9) by using an optimal solution (v, u) of (7). And, in a similar manner, we can provide a definition of the cross-efficiencies in the context of the assessment of efficiency with directional distance functions.