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Optimizing bounds on security prices in incomplete markets. Does stochastic volatility specification matter?

Naroa Marroquín-Martínez [a], Manuel Moreno [b,*]

[a] Department of Applied Economics III (Econometrics and Statistics), University of the Basque Country, 48015 Bilbao, Spain

[b] Department of Economic Analysis and Finance, University of Castilla La-Mancha, Cobertizo de San Pedro Martir, s/n, 45071 Toledo, Spain

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ABSTRACT

We extend and generalize some results on bounding security prices under two stochastic volatility models that provide closed-form expressions for option prices. In detail, we compute analytical expressions for benchmark and standard good-deal bounds. For both models, our findings show that our benchmark results generate much tighter bounds. A deep analysis of the properties of option prices and bounds involving a sensitivity analysis and analytical derivation of Greeks for both option prices and bounds is also presented. These results provide strong practical applications taking into account the relevance of pricing and hedging strategies for traders, financial institutions, and risk managers.

1. Introduction

Under the assumption of complete markets, every contingent claim can be replicated by a portfolio formed of the underlying basic assets of the market. In this case, the equivalent martingale measure and the market price of risk are unique and, then, the price of any security is uniquely determined by this martingale measure. Nevertheless, in the real world this situation barely happens and there can be (infinitely) many equivalent martingale measures.

When there are sources of risk that are not directly traded (such as stochastic volatility, jumps or weather) the assumption of complete market fails. Staum (2008) surveys many approaches to pricing and hedging derivative securities under incomplete markets. As there will not exist a unique martingale measure, there will exist infinitely many arbitrage-free price processes for a certain financial security. Then, it can be interesting to derive no-arbitrage bounds on asset prices and obtain a no-arbitrage interval, where the price of the asset should lie.

Several papers have computed bounds on option prices. For instance, Basso and Pianca (2001) consider a state-preference approach and provide lower and upper bounds for European option prices by solving a non-linear optimization problem. No-arbitrage bounds can also be computed using information about prices of other options on the same underlying asset, see Bertsimas and Popescu (2002) or d'Aspremont and El Ghaoui (2006), among others. Working in discrete-time, Reynaerts et al. (2006) focus in Cox et al. (1979) model with daily time step and derive bounds on prices for arithmetic Asian options with discrete sampling. These bounds can also be obtained assuming an incomplete knowledge of the underlying price distribution. For example, Zuluaga et al. (2009) derive closed-form semi-parametric bounds for the payoff of a European call option, given up to third-order statistical moments for the underlying asset distribution at maturity.

Considering incomplete markets, Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000) try to find no-arbitrage bounds on prices as tight as possible using the stochastic discount factor (SDF) as starting point.[1] Then, both papers restrict the pricing kernel to derive these bounds. Extracting a paragraph from Franke et al. (2007, p. 215), "Cochrane and Saá-Requejo (2000) show that the option price can be bounded by limiting the variance of the pricing kernel. In similar vein, Bernardo and Ledoit (2000) show that the option price can be bounded by limiting the convexity of the pricing kernel." The intuition behind these two papers is that investors will choose a trading asset price according to some optimality criterion. As mentioned in Pinar et al. (2010, p. 771), "in Cochrane and Saá-Requejo (2000), the absence of arbitrage is replaced by the concept of a good deal, defined as an investment with a high Sharpe ratio. While they do not use the term "good-deal", Bernardo and Ledoit (2000) replace the high Sharpe ratio by the gain-loss ratio."

* Corresponding author. Tel.: +34 925 26 88 00; fax: +34 925 26 88 01.
E-mail addresses: naroa.marroquin@hotmail.com (N. Marroquín-Martínez), manuel.moreno@uclm.es (M. Moreno).

[1] The SDF was introduced in Hansen and Jagannathan (1991) who demonstrated that a bound on the maximum available Sharpe ratio is equivalent to a bound on the volatility of the admissible SDF.

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In more detail, Bernardo and Ledoit (2000) analyze different investment opportunities using the *gain-loss* ratio as a performance measure using a benchmark or reference asset pricing model. They demonstrate that a high *gain-loss* ratio is related to SDF's that are "specially" far from the benchmark SDF. In this way, an appropriate benchmark SDF can tighten the no-arbitrage bounds and the corresponding no-arbitrage interval. Inspired by this paper, Pinar et al. (2010) apply linear programming to price and hedge contingent claims in a multi-period setting and propose an optimality criterion, the "λ gain-loss ratio", that treats asymmetrically gains and losses. The pricing bounds obtained are tighter than the no-arbitrage ones and, as expected, converge to the no-arbitrage ones as the gain-loss preference parameter tends to infinity. These authors also show that a unique claim price may be found for a limiting case of the risk aversion parameter.

Alternatively, Cochrane and Saá-Requejo (2000) also measure the attractiveness of an investment but using the Sharpe ratio and suggested to rule out usual arbitrage opportunities with too high Sharpe ratios. Thus, they obtain tighter price bounds that are named *benchmark good-deal* (BGD) bounds. Several authors have dealt with this approach and proposed different methodologies to compute this type of bounds. For instance, Cerný and Hodges (2002) present the theory of good-deal pricing in financial markets and shows that "any such technique can be seen as a generalization of no-arbitrage pricing and that, with a little bit of care, it will contain the no-arbitrage and the representative agent equilibrium as the two opposite ends of a spectrum of possible no-good-deal equilibrium restrictions." In a related paper, Cerný (2003) replaces the Sharpe ratio (connected to quadratic utility) with a generalized Sharpe ratio based on an arbitrary increasing smooth utility function and shows that "for Itô processes Cochrane and Saá-Requejo (2000) bounds are invariant to the choice of the utility function, and that in the limit they tend to a unique price determined by the minimal martingale measure".

Björk and Slinko (2006) extend the setting in Cochrane and Saá-Requejo (2000) by studying arbitrage-free good-deal pricing bounds for derivative assets and presented results for the Merton-jump diffusion model. Additionally, they derive extended Hansen–Jagannathan bounds for the Sharpe ratio process in the point process setting. Albanese and Tompaidis (2008) consider the good-deal pricing literature and perform a dynamic risk-reward analysis for a type of time-based hedging strategies in the presence of transaction costs. Pinar (2008) uses an arbitrage-adjusted Sharpe-ratio criterion and convex optimization and provides bounds on contingent claim prices that are tighter than the no-arbitrage ones. Finally, Bondarenko and Longarela (2009) present asset price bounds as the result of an optimization problem over a set of admissible SDF's. They consider the option pricing model presented in Heston (1993) and assume certain limits for the volatility risk premium. They derive closed-form solutions for the BGD bounds and for a particular case, standard good-deal (GD) bounds, showing that the former are much tighter than the latter.

Continuing with this research area, our paper focuses on computing and analyzing BGD and GD bounds for different asset prices under two stochastic volatility option pricing models, that introduced in Heston (1993) and an extension of that posited in Schöbel and Zhu (1999). In this way, we can get an insight into the effects of different specifications for the stock volatility process on the aforementioned bounds.

Heston (1993) generalizes the classical model for stock prices presented in Black and Scholes (1973) allowing the stock volatility to follow a "square-root" (CIR-type) stochastic process as presented in Cox et al. (1985). Additionally, Schöbel and Zhu (1999) extended the stochastic volatility model of Stein and Stein (1991), where the stock volatility follows an Ornstein–Uhlenbeck process. They allow correlation to exist between the underlying stock returns and the instantaneous volatility and found a closed-form expression for option prices.

As these two models deal with stochastic volatility, markets are incomplete. However, both provide unique closed-form expressions for the prices of certain securities assuming a certain functional form of the market price(s) of risk of the corresponding factor(s). In fact, each functional form is associated to a martingale measure and, thus, to a price for the security.

This paper contributes to the existing literature in three ways: firstly, we analyze deeply Bondarenko and Longarela (2009) and fix different errors in their numerical analysis. One of our main results is that, now, the difference between GD and BGD bounds is stronger than that previously reported by these authors. Secondly, we extend the Schöbel and Zhu (1999) model allowing the market price of risk of the volatility to be different from zero. In this extended model, with no new mathematical ideas, we also obtain analytical expressions for option prices and their bounds. Numerical illustrations are shown for all the bounds obtained.

Our final contribution is that, for both models, extensive sensitivity analysis are carried out studying how changes in the models' parameters affect prices and bounds. Additionally, we also implement a hedging analysis by computing several Greeks for prices and bounds. Computation of these Greeks is relevant because, as shown in Carr (2001), these amounts can be interpreted as the values of certain quantoed contingent claims. Besides, as this author states, "this interpretation allows one to transfer intuitions regarding values to these Greeks and to apply any valuation methodology to determine them".

The structure of the paper is as follows. Section 2 describes the theoretical framework that is needed to find the bounds on option prices. Stochastic volatility models are presented in Section 3. Section 4 derives analytical expressions for option prices and their bounds under these models. A deep analysis of the properties of option prices and bounds involving a sensitivity analysis and derivation of Greeks for both option prices and bounds is included in Section 5. Finally, Section 6 summarizes the main findings and conclusions.

2. Theoretical framework

We present now our theoretical framework. Consider a probability space (Ω, F, P) with the corresponding filtration $\{F_t\}_{t \geqslant 0}$. Assume that we have a bond that pays the risk-free rate r_t, a risky asset S_t (stock), and one (non-tradable) state variable V_t. Let (W_t^s, W_t^v) be two standard and independent Brownian motions and let $h_t = (h_t^s, h_t^v)$ be an adapted two dimensional process, which satisfies the Novikov condition. Departing from the probability measure P, we define the measure Q via the Radon–Nikodim derivative, that is

$$\frac{dQ}{dP} = \xi_T$$

where for all t

$$\xi_t = \exp\left[-\int_0^t h_u \, dW_u - \frac{1}{2}\int_0^t \|h_u\|^2 \, du\right]$$

and ξ is a P-martingale with expected value equal to one. The SDF process is defined as

$$\Lambda_t = B_t \xi_t$$

where $B_t = \exp\left(-\int_0^t r_u du\right)$. Applying Itô's lemma, we get that

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t \, dt - h_t' \, dW_t$$

We can define the benchmark model in terms of the vector process $h_t^* = (h_t^{*s}, h_t^{*v})$ with the corresponding martingale measure, Q^*, and