European Journal of Operational Research 225 (2013) 472–478

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Generalization of the weighted voting method using penalty functions constructed via faithful restricted dissimilarity functions

H. Bustince [a,*], A. Jurio [a], A. Pradera [b], R. Mesiar [c,d], G. Beliakov [e]

[a] Departamento de Automatica y Computacion, Universidad Publica de Navarra, Campus Arrosadia s/n, 31006 Pamplona, Spain
[b] Departamento de Ciencias de la Computación, Universidad Rey Juan Carlos, 28933 Móstoles, Madrid, Spain
[c] Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Radlinského 11, 81368 Bratislava, Slovakia
[d] Institute for Research and Applications of Fuzzy Modelling, University of Ostrava, 70103 Ostrava, Czech Republic
[e] School of Information Technology, Deakin University, 221 Burwood Hwy, Burwood 3125, Australia

ARTICLE INFO

Article history:
Received 7 October 2011
Accepted 6 October 2012
Available online 17 October 2012

Keywords:
Restricted dissimilarity function
Penalty function
Selection process
Weighted voting method

ABSTRACT

In this paper we present a generalization of the weighted voting method used in the exploitation phase of decision making problems represented by preference relations. For each row of the preference relation we take the aggregation function (from a given set) that provides the value which is the least dissimilar with all the elements in that row. Such a value is obtained by means of the selected penalty function. The relation between the concepts of penalty function and dissimilarity has prompted us to study a construction method for penalty functions from the well-known restricted dissimilarity functions. The development of this method has led us to consider under which conditions restricted dissimilarity functions are faithful. We present a characterization theorem of such functions using automorphisms. Finally, we also consider under which conditions we can build penalty functions from Kolmogoroff and Nagumo aggregation functions. In this setting, we propose a new generalization of the weighted voting method in terms of one single variable functions. We conclude with a real, illustrative medical case, conclusions and future research lines.

1. Introduction

Consider:

- a system of n numerical inputs $(x_1, \ldots, x_n)$,
- a system of q aggregation functions $(M_1, \ldots, M_q)$ (see [1,7]),
- the system of values $(y_1 = \overset{n}{\underset{i=1}{M_1}} x_i, \ldots, y_q = \overset{n}{\underset{i=1}{M_q}} x_i)$ (each of them obtained by aggregating n inputs with different aggregation functions).

The notion of a *penalty based aggregation function* (see [8,9,25]) allows us to determine the aggregated value y_j which is the least dissimilar to the set of inputs $\{x_1, \ldots, x_n\}$.

One of the most widely used methods in decision making is the weighted voting method (see [18–20]). It consists in calculating the weighted arithmetic mean of the elements in each of the rows of the preference relation provided by the expert (see [10]), and selecting the alternative associated with the row with largest value as the solution. This description of the method poses the question of whether the use of the weighted arithmetic mean is the best option for all possible scenarios.

This consideration has led us to the objective of this paper: to use the concept of penalty function to determine, in a decision making problem, which aggregation function to apply to each row of the considered preference relation so that the result (output) is the least dissimilar to the elements (inputs) in that row. Moreover, the new method should allow us, under suitable conditions, to recover the weighted voting method.

In the notion of penalty function, the concept of dissimilarity plays a very important role. It is known that in some applications, such as image processing, the so-called *restricted dissimilarity functions* (see [4]) are used to measure how dissimilar two areas (objects) are. These considerations lead us to use these (restricted) dissimilarities to build penalty functions since, as we will prove in this work, restricted dissimilarity function can be easily built by means of automorphisms (see [3,15]), whereas this is not generally the case for dissimilarity functions.

The aggregation, in a proper way, of convex or quasi-convex dissimilarity functions can generate penalty functions. However, the difficulty to characterize the convexity property (and the quasi-convexity property) for dissimilarity functions [25] as well as for restricted dissimilarity functions, on one hand, and the theoretical developments in [8,25] on the other hand, have led us to study

* Corresponding author. Tel.: +34 948169254.
E-mail addresses: bustince@unavarra.es (H. Bustince), aranzazu.jurio@unavarra.es (A. Jurio), ana.pradera@urjc.es (A. Pradera), mesiar@math.sk (R. Mesiar), gleb@deakin.edu.au (G. Beliakov).

0377-2217/$ - see front matter
http://dx.doi.org/10.1016/j.ejor.2012.10.009

convex faithful restricted dissimilarity functions. We also know that we can construct distances in the sense of Liu between fuzzy sets (see [24,13,5,4]) by properly aggregating restricted dissimilarity functions. This fact has led us to build penalty functions aggregating, in a proper way, faithful restricted dissimilarity functions. In particular, we are going to use weighted quasi-arithmetic aggregation functions (see [8]). Such aggregations allow:

1. To build penalty functions using one single variable functions and faithful restricted dissimilarity functions.
2. To recover the weighted voting method in decision making.

We have organized this work as follows: In the preliminaries section we introduce the basic concepts that are needed for subsequent developments. In Section 3 we study the relations between restricted dissimilarity functions, negations and automorphisms. In Section 4 we introduce the concept of faithful restricted dissimilarity function and we present a characterization theorem by means of automorphisms. We also present a construction method of convex faithful restricted dissimilarity functions. Then in Section 5 we study the construction of penalty functions from convex faithful restricted dissimilarity functions and we analyze such constructions when we use Kolmogoroff and Nagumo aggregation functions. In Section 6 we apply our theoretical developments to decision making and we present an algorithm for the exploitation phase that generalizes the weighted voting method. We present two easier versions of the algorithm depending on the type of used functions. We also present a decision making problem for selecting medication for a patient suffering from hypertension. We finish with conclusions and references.

2. Preliminary definitions

Zadeh [29] introduced the fuzzy set theory in 1965. A fuzzy set A on the universe $U \neq \emptyset$ is a mapping A: $U \to [0,1]$. We will denote by $\mathcal{F}(U)$ the set of all fuzzy sets defined on the finite, non-empty referential set U ($Cardinal(U) = n$). A *fuzzy negation* is a non-increasing function N: $[0,1] \to [0,1]$ such that $N(0) = 1$ and $N(1) = 0$. If the mapping N is strictly decreasing, then N is called a strict negation. A strong negation is a non-increasing function N: $[0,1] \to [0,1]$ which is involutive; that is, $N(N(x)) = x$ for any $x \in [0,1]$. Recall that a function N: $[0,1] \to [0,1]$ is a strong negation if and only if there exists an automorphism (a strictly increasing bijection) φ: $[0,1] \to [0,1]$ such that $N = N_\varphi$, where $N_\varphi(x) = \varphi^{-1}(1-\varphi(x))$ for all $x \in [0,1]$. Due to their definition, strong negations are continuous and strictly decreasing functions, and satisfy the boundary conditions $N(0) = 1$ and $N(1) = 0$.

2.1. Aggregation functions

Definition 1 [1,6,7,16]. A mapping M:$[a,b]^n \to [a,b]$ is an aggregation function if it is monotone non-decreasing in each of its components and satisfies $M(a, a, \ldots, a = a$ and $M(b, b, \ldots, b) = b$.

In general we will take $[a,b] = [0,1]$.

Definition 2 [15]. An aggregation function M is called averaging or a mean if

$$\min(\mathbf{x}) = \min(x_1,\ldots,x_n) \leqslant M(x_1,\ldots,x_n) \leqslant \max(x_1,\ldots,x_n) = \max(\mathbf{x})$$

for all $(x_1, \ldots, x_n) \in [a,b]^n$.

Any averaging aggregation function is idempotent, and also the converse is true.

2.2. Penalty functions

We need to measure the disagreement or dissimilarity between an input $\mathbf{x} = (x_1, \ldots, x_n)$ and the corresponding output y. We are going to call such a measure a *penalty function*. Our aim is to find for a given penalty function the aggregation function that minimizes it.

Definition 3. [2,8] A penalty function is a mapping $P : [a,b]^{n+1} \to \mathcal{R}^+ = [0,\infty]$ such that:

(1) $P(\mathbf{x},y) = 0$ if $x_i = y$ for all $i = 1, \ldots, n$;
(2) $P(\mathbf{x},y)$ is quasi-convex in y for any $\mathbf{x}$; that is, for each fixed $\mathbf{x} \in [a,b]^n$ the inequality

$$P(\mathbf{x},\lambda y_1 + (1-\lambda)y_2) \leqslant \max(P(\mathbf{x},y_1),P(\mathbf{x},y_2))$$

holds for any $\lambda \in [0,1]$ and any $y_1, y_2 \in [a,b]$.

Let P be a penalty function. We call *penalty based function* [8] (or function based on the penalty function P) the mapping

$$f(\mathbf{x}) = \arg\min_y P(\mathbf{x},y),$$

if y is the only minimizer and $y = \frac{c+d}{2}$ if the set of minimizers is given by the interval $[c,d]$.

A penalty based function is always idempotent, but not necessarily monotone (see [8]). The following theorem states that also the converse holds when referring to idempotency.

Theorem 1 [8]. *Any idempotent function can be represented as a penalty based function in the sense of Definition 3.*

3. Restricted dissimilarity functions, negations and automorphisms

In [4] the concept of *restricted dissimilarity function* is introduced from the concept of equivalence function defined by Fodor and Roubens in [15]. The conditions that are demanded to restricted dissimilarity functions allow to develop construction methods which are much more general than those existing for dissimilarity functions and moreover, the newly required conditions make that these measures may be applied in more fields, such as image processing.

In this section we present some construction methods of restricted dissimilarity functions from automorphisms. We also study the construction of automorphisms and negations from restricted dissimilarity functions.

Definition 4. [4]. A mapping d_R: $[0,1]^2 \to [0,1]$ is a restricted dissimilarity function if:

(1) $d_R(x,y) = d_R(y,x)$ for every $x,y \in [0,1]$;
(2) $d_R(x,y) = 1$ if and only if $x = 0$ and $y = 1$ or $x = 1$ and $y = 0$; that is, if and only if $\{x,y\} = \{0,1\}$;
(3) $d_R(x,y) = 0$ if and only if $x = y$;
(4) For any $x, y, z \in [0,1]$, if $x \leqslant y \leqslant z$, then $d_R(x,y) \leqslant d_R(x,z)$ and $d_R(y,z) \leqslant d_R(x,z)$.

We will say that d_R is a strict restricted dissimilarity function if for any $x, y, z \in [0,1]$, if $x < y < z$, then $d_R(x,y) < d_R(x,z)$ and $d_R(y,z) < d_R(x,z)$.

Example 1. The mapping $d_R(x,y) = |x-y|$ provides a simple example of restricted dissimilarity function which is strict. On the other hand, as an example of non-strict restricted dissimilarity function we can present the following. Take $c \in]0,1[$. Then