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Decision Support

Truth-telling and Nash equilibria in minimum cost spanning tree models

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ABSTRACT

In this paper we consider the minimum cost spanning tree model. We assume that a central planner aims at implementing a minimum cost spanning tree not knowing the true link costs. The central planner sets up a game where agents announce link costs, a tree is chosen and costs are allocated according to the rules of the game. We characterize ways of allocating costs such that true announcements constitute Nash equilibria both in case of full and incomplete information. In particular, we find that the Shapley rule based on the irreducible cost matrix is consistent with truthful announcements while a series of other well-known rules (such as the Bird-rule, Serial Equal Split, and the Proportional rule) are not.

1. Introduction

Recently, economists have shown a growing interest in networks and the literature is becoming rich on various issues and models, see, e.g. Goyal (2007) and Jackson (2008). In the present paper we consider the relation between cost allocation and efficient network structure within the classical minimum cost spanning tree (MCST) model. Here, a group of agents is to be connected to a source (supplier) in the least costly way and face the problem of sharing the cost of the efficient network, see, e.g. Sharkey (1995) – practical examples include district heating, computer network using a common server, cable tv, chain stores using a common warehouse, etc.

While the literature typically contains axiomatic analysis and comparisons of different cost sharing methods there has been less emphasis on strategic issues concerning practical implementation. Clearly, agents can have private information about the cost structure. This information they can use strategically to lower their own cost at the expense of a loss in social efficiency.

It is well-known that in general we cannot find budget balanced, incentive compatible and efficient cost allocation mechanisms (Green and Laffont, 1977), but under special circumstances such mechanisms can in fact be constructed, see, e.g. Jackson and Moulin (1992), Schmeidler and Tauman (1994), Young (1998), and Moulin and Shenker (2001). For social choice rules Maskin (1999) and Dasgupta et al. (1979) show that implementability and monotonicity are equivalent. We consider monotonicity of cost allocation rules and present compatible results for the MCST model.

Efficient implementation in general connection networks has also been the focus of a strand of literature in Computer Science, see, e.g. Chen et al. (2010) and Kumar and Juarez (2011). Typically, different so-called cost allocation protocols are analyzed with respect to inefficiency measures such as price of anarchy (POA) and price of stability (POS).

In the specific context of the MCST model, implementation has been analyzed in a few recent papers; Bergantinos and Lorenzo (2004, 2005) and Bergantinos and Vidal-Puga (2010). All three papers consider existence and properties of Nash equilibria and subgame perfect Nash equilibria of non-cooperative sequential bargaining procedures. The first two papers study a real life allocation problem where agents sequentially join an existing network along the lines of the Prim algorithm (Prim, 1957). The latter paper shows that another Prim-like procedure, where agents announce their willingness to pay for other agents to connect to the source, leads to a unique subgame perfect Nash equilibrium in which costs are allocated corresponding to the use of the Shapley value on the related irreducible cost matrix (dubbed the Folk-solution in Bogomolnaia and Moulin (2010) and further analyzed in Bergantinos and Vidal-Puga (2007, 2009) and Hougaard et al. (2010)).

Finally, Ozsoy (2007) and Gomez-Rua and Vidal-Puga (2011) take an axiomatic approach to manipulation using properties of merge-proofness[2] to characterize the Bird rule.

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[2] Intuitively, merge-proofness means that coalitions cannot gain by merging into a single agent, forming their own network within the coalition, and connect as a group to the rest of the network.

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In the present paper, Section 2 briefly reviews the MCST model and Section 3 introduces a non-cooperative game form involving a planner who does not know the link costs and a set of agents who all know the link costs. First the planner announces the rules of the game being an allocation rule and an estimation rule. Then agents announce the link costs, which in turn are used to estimate a cost matrix and its related set of MCSTs. A particular MCST is selected at random and the realized (true) link costs are shared between the agents according to the announced allocation rule.

Compared to the Prim-like sequential mechanisms mentioned above our approach is different since it is a simple one-shot game, which is not based on any algorithm for finding the MCST. Moreover, agents' announcements do not directly influence their cost shares since these are determined by the realized (true) costs along the chosen spanning tree.

In Section 4 our main result is established: announcing the true link costs constitutes a Nash equilibrium if and only if the associated allocation rule is monotonic (in the sense that cost shares are weakly increasing in the irreducible cost matrix). Consequently, monotonic allocation rules such as the Shapley rule (on the irreducible cost matrix) and the Equal Split rule will both result in truth-telling Nash equilibria where the planner can implement the true MCST. However, well-known rules such as the Bird rule and the Proportional rule fail to satisfy monotonicity.

In Section 5 we consider an incomplete information version of the game along the lines of Jackson (1991). We show that truth-telling remains an equilibrium for monotonic allocation rules. Section 6 closes with some final remarks.

2. The MCST model

Recall the minimum cost spanning tree model (see, e.g. Sharkey, 1995). Networks, where a source supplies agents with some homogeneous good, are considered. Let 0 be the source and let $N = \{1, \ldots, n\}$ be the set of agents. A network g over $N^0 = \{0\} \cup N$ is a set of unordered pairs ab, where $a, b \in N^0$. Let $N^0(2)$ be the set of unordered pairs and let $G^0 = \{g | g \subset N^0(2)\}$ be the set of all networks over N^0.

In a network g two agents a and b are *connected* if and only if there is a path $i_1i_2, i_2i_3, \ldots, i_{m-1}i_m$ such that $i_hi_{h+1} \in g$ for $1 \leqslant h \leqslant m-1$ where $i_1 = a$ and $i_m = b$. A network g is *connected* if a and b are connected for all $a, b \in N^0$. A path is a *cycle* if it starts and ends with the same agent. A network is a *tree* if it contains no cycles. A *spanning tree* is a tree where all agents in N^0 are connected. There are $(n+1)^{n-1}$ spanning trees.

For every pair of agents $ab \in N^0(2)$ there is a cost $k_{ab} \geqslant 0$ associated with the link between a and b. This cost may be interpreted as the cost of establishing the link ab. For N^0, let the $(n+1) \times (n+1)$-matrix K be the cost matrix (where $k_{aa} = 0$ for all $a \in N^0$). Note that $k_{ab} = k_{ba}$ since the network is undirected so K is symmetric around the diagonal. An *allocation problem* is a set of agents and a cost matrix (N, K).

For a spanning tree p, let $v(N,K,p) = \sum_{ab \in p} k_{ab}$ be the total cost of p. A *minimum cost spanning tree* (MCST) is a spanning tree p such that $v(N, K, p) \leqslant v(N, K, q)$ for every spanning tree q. For every allocation problem (N, K) there exists a MCST because the number of spanning trees is finite. Let $v(N, K)$ denote the minimal cost so there exists a spanning tree p such that $v(N, K, p) = v(N, K)$ and $v(N, K, q) \geqslant v(N, K)$ for every spanning tree q. If all costs k_{ab} are different then there is a unique MCST, but in general there can be several MCSTs. Indeed, if all costs k_{ab} are equal, then every spanning tree is a MCST. Let $T(N, K)$ be the set of MCSTs.

2.1. Irreducible matrices

For two matrices K and K' the matrix K is smaller than K' if and only if $k_{ab} \leqslant k'_{ab}$ for all $a, b \in N^0$. The *irreducible* matrix $C(K)$ for a cost matrix K is the smallest matrix C such that $v(N, C) = v(N, K)$ and $c_{ab} \leqslant k_{ab}$ for all $a, b \in N^0$, see, e.g. Bird (1976) and Aarts and Driessen (1993). For a cost matrix K and a spanning tree p the irreducible matrix $C(K, p)$ is defined as follows: For every $a, b \in N^0$, let p_{ab} be the unique path in p from a to b, then $c_{ab} = \max_{ij \in p_{ab}} \{k_{ij}\}$. It is known that if p^* is a MCST, then $C(K) = C(K, p^*)$. Hence if p^* and q^* are MCSTs, then $C(K, p^*) = C(K, q^*) = C(K)$, and if p^* is a MCST and p is spanning tree, then $C(K, p^*)$ is smaller than $C(K, p)$.

2.2. Allocation rules

Let Γ be the set of allocation problems and their spanning trees so $(N, K, p) \in \Gamma$ if and only if (N, K) is an allocation problem and p is a spanning tree for (N, K). An allocation rule $\phi : \Gamma \to \mathbb{R}^N$ maps an allocation problem (N, K) and a spanning tree p to an n-dimensional vector of cost shares $\phi(N, K, p) = (\phi_1(N, K, p), \ldots, \phi_n(N, K, p))$.

Only allocation rules that are *budget balanced*, *reductionist*[3] and *continuous* are considered.

- *Budget-balance:* $\sum_{a \in N} \phi_a(N,K,p) = v(N,K)$ for all $p \in T(N, K)$, so cost shares add up to the total cost of the MCST.
- *Reductionist:* $\phi_a(N, K, p) = \phi_a(N, C(K, p), p)$ for all spanning trees p, so cost shares depend on the irreducible matrix of the chosen spanning tree.
- *Continuity:* $\phi(N, K, p)$ is continuous in K.

Budget-balance and continuity are standard properties of allocation rules. Budget-balance implies that exactly the cost of every spanning tree is allocated. The reason why reductionist rules are considered is closely related to our implementation setting where only costs along the realized spanning tree are observed by the planner. Thus, by relying on the construction of the irreducible cost matrix only the revealed cost information is used.

For a spanning tree p, let $\delta(a, b, p)$ be the unique neighbor of a in the path p_{ab} from a to b in p. Some examples of (continuous, budget-balanced and reductionist) allocation rules include:

- *The Equal Split Rule:* For all $a \in N$

$$\phi_a^E(N,K,p) = \frac{v(N, C(K,p))}{n}.$$

- *The Bird Rule:* For all $a \in N$

$$\phi_a^B(N,K,p) = c_{ab}(K,p)$$

where $b = \delta(a, 0, p)$.

- *The Proportional Rule:* For all $a \in N$

$$\phi_a^P(N,K,p) = \frac{c_{0a}(K,p)}{\sum_{b \in N} c_{0b}(K,p)} v(N, C(k,p)).$$

- *The Folk Rule:* For all $a \in N$

$$\phi_a^S(N,K,p) = \sum_{S \subset N: a \in S} \frac{(s-1)!(n-s)!}{n!} (v(S, C_S) - v(S \setminus a, C_{S \setminus a}))$$

where s is the number of agents in S, C_S is the projection of $C(K, p)$ on $\{0\} \cup S$.[4]

[3] Allocation rules based on the irreducible matrix are denoted *reductionist* rules in Bogomolnaia and Moulin (2010).

[4] The Folk rule for (N, K) coincides with the Shapley value for $(N, C(K))$, see Bergantinos and Vidal-Puga (2007, 2009) and Bogomolnaia and Moulin (2010). Furthermore, the Folk rule also coincides with the Equal Remaining Obligation Rule for $(N, C(K))$ defined in Feltkamp et al. (1994).