European Journal of Operational Research 222 (2012) 663–672

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Interfaces with Other Disciplines

Robust risk management

Apostolos Fertis *, Michel Baes, Hans-Jakob Lüthi

Institute for Operations Research (IFOR), Eidgenössische Technische Hochschule Zürich, Rämistrasse 101, HG G 21.3, 8092 Zürich, Switzerland

ARTICLE INFO

Article history:
Received 12 May 2011
Accepted 19 March 2012
Available online 28 March 2012

Keywords:
Convex programming
Robust optimization
Risk management

ABSTRACT

Estimating the probabilities by which different events might occur is usually a delicate task, subject to many sources of inaccuracies. Moreover, these probabilities can change over time, leading to a very difficult evaluation of the risk induced by any particular decision. Given a set of probability measures and a set of nominal risk measures, we define in this paper the concept of robust risk measure as the worst possible of our risks when each of our probability measures is likely to occur. We study how some properties of this new object can be related with those of our nominal risk measures, such as convexity or coherence. We introduce a robust version of the Conditional Value-at-Risk (CVaR) and of entropy-based risk measures. We show how to compute and optimize the Robust CVaR using convex duality methods and illustrate its behavior using data from the New York Stock Exchange and from the NASDAQ between 2005 and 2010.

1. Introduction

In quantitative risk management, risk measures are used to determine a preferential order among financial positions with random outcome. Each financial position is seen as a random variable that maps each state of nature ω to a real number. This number corresponds to the reward ensured by the financial position when the state ω occurs. Risk measures are designed to take into account the trade-off between the magnitudes of the values that a position can take, and the risk or variability in these values. Mathematically, they are mappings of a space of random variables to the extended real line. The choice of the risk measure determines the investment risk profile.

A portfolio is a linear combination of some available assets, each characterized by a cost and a random variable representing their income, in the limits of a fixed budget. Markowitz (1952) defines the risk of a portfolio as a weighted sum of its expected return and its variance. The ratio between the two weights determines the risk profile of the investor. Since Markowitz's breakthrough paper, many other risk measures have been introduced. Most notably, Value-at-Risk (VaR), a quantile of the position's probability distribution, has been extensively used (RiskMetrics, 1995). However, VaR has been criticized for not detecting unfavorable behavior in the tails of the probability distribution (Donnelly and Embrechts, 2010). This observation triggered the introduction of classes of risk measures that satisfy some desirable properties. For instance, the class of convex risk measures (Föllmer and Schied, 2002) gathers monotone and convex mappings that satisfy a translation invariance property (see in Section 2). Every convex risk measure can be expressed as the conjugate of some "penalty" function defined in a space of signed measures. This representation can be used to compute optimal portfolios and to assess their value of flexibility (Lüthi and Doege, 2005). Moreover, convex risks are closely connected with the concept of the optimized certainty equivalent introduced in Ben-Tal and Teboulle (2007). Coherent risk measures (Artzner et al., 1999) forms the subclass of convex risk measures that are positively homogenous (see Section 2). They can be expressed as the worst-case expectation of the portfolio outcome when the probability measure of the assets returns varies in some uncertainty set (Artzner et al., 1999). For instance, Conditional Value-at-Risk (CVaR), that is, the expected value of a portfolio if its loss lies beyond some quantile of its distribution, is such a coherent risk measure. The composition of a portfolio optimal with respect to its CVaR can be computed using the dual representation of this risk measure (Rockafellar and Uryasev, 2000; Shapiro et al., 2009).

Due to the intrinsic uncertainty of the environment they describe, it can happen that the data defining a problem is not known exactly. As a result, it is possible that the optimal solution computed for the erroneous problem we have is far from optimal, or not even feasible, for the actual problem. Robust optimization is now increasingly used to tackle his issue. It considers that the actual data of a problem belongs to a predefined uncertainty set $\mathcal{S}$, then assigns to every feasible point the worse objective value among all the problems with data in $\mathcal{S}$. The optimal point it returns is then the feasible point with the best of those worse values, and is thereby immune to data uncertainties. In linear optimization problems, Soyster (1973) considers box-type uncertainty sets and (Ben-Tal and Nemirovski, 1998, 1999, 2000) ellipsoidal ones. In linear and in mixed integer optimization, problems with budgeted

* Corresponding author. Tel.: +41 44 632 4031.
E-mail address: afertis@ifor.math.ethz.ch (A. Fertis).

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http://dx.doi.org/10.1016/j.ejor.2012.03.036

uncertainty sets (Ben-Tal et al., 2009) for their constraints are efficiently solved in Bertsimas and Sim (2003, 2004). Interestingly, minimizing the coherent risk measure of an affine combination of random variables can be reformulated as a robust optimization problem; an explicit description of the uncertainty set is given in Bertsimas and Brown (2009).

Not only the data of the problem can be subjected to errors, but also the probability distribution model for the random positions, as it is constructed, among other sources, from possibly corrupted historical data. Several approaches deal with this issue. A first possibility consists in defining a class of parameterized probability distributions, among which the actual one is determined by standard parameter estimation procedures. Moreover, these procedures can yield confidence intervals, which can be used as uncertainty sets in a robust optimization framework (Bertsimas and Pachamanova, 2008). Robust solutions are unavoidably conservative: due to the typically infinite number of extra constraints, the obtained return is often much lower than the non-robust return. Among other techniques to tackle this issue, let us point the one developed in Zymler et al. (2011). There, the set of constraints ensures a certain minimal return when the probability parameters belong to a certain small set, and another minimal return when they belong to another, larger, set.

In this paper, we consider the problem of assessing risk when the probability measure driving the underlying random process is not known exactly, but resides in some uncertainty set, called here the scenarios set. Instead of using a single risk measure for our whole problem, we have one risk measure per probability function from our scenario set. In other words, we use a family of risk measures, each indexed by a probability measure from the scenarios set. We define our robust risk measure as an appropriate combination of them. Some of their properties, such as convexity and coherence can be shown to be transferred to our robust risk measure (see Propositions 2.1 and 2.2). Our definition is then particularized to define a robust counterpart to Conditional Value-at-Risk (CVaR) and to entropy-based risks in the context of two-stage structured uncertainty sets (see SubSection 3.1 and Section 4 for precise definitions). We also provide efficient algorithms to compute these risks. Robust CVaR has been successfully used in hydro-electric pumped storage plant management (Fertis and Abegg, 2010), and has been connected with regularization in portfolio optimization (Fertis et al., 2011).

Special cases for the Robust CVaR under two-stage structured uncertainty sets, namely the cases when the probability measures are discrete or when the probability measure uncertainty set is the whole set of probability measures on the considered space, have been already studied (Zhu and Fukushima, 2009). In this paper, we deal with continuous or discrete probability measures, and generic norm-bounded uncertainty sets. In the case of Robust CVaR for continuous distributions, our result enables portfolio optimization through the stochastic average approximation method, which discretizes the sample space (Shapiro et al., 2009). Robust risk measures with different probability distribution uncertainty sets have been considered in the past as well. The worst-case CVaR when certain moments of the assets' probability distribution are known can be expressed as a finite-dimensional robust optimization problem, and can be efficiently computed if the distribution is discrete or continuous of a special kind (Natarajan et al., 2009). The worst-case CVaR can be computed through linear optimization when the probability distribution is required to replicate the prices of some European put and call options on the assets (Jabbour et al., 2008). Portfolio optimization according to worst-case CVaR when the probability distribution uncertainty set is defined through the Kantorovich distance has been proved to lead to the uniform investment strategy (Pflug et al., 2012). The worst-case CVaR under certain moment information has been considered in the framework of chance constrained optimization, and has been compared to the worst-case VaR under certain moment information (Zymler et al., in press).

The paper is structured as follows:

- In Section 2, we define the robust risk measure with reference to a family of nominal risk measures and an uncertainty set for the probability measure that drives the random process. We investigate the structure of the robust risk measure when the family of nominal risk measures contains convex or coherent risks.
- In Section 3, we define Robust CVaR, as the robust risk measure corresponding to CVaR. When the scenario set is structured in two stages, and uncertainty is limited in the second-stage probability distribution, we show how to compute the Robust CVaR of a position, and how to compute portfolios that optimize the Robust CVaR. The complexity of the proposed algorithms is almost the same as the complexity of the corresponding algorithms for CVaR.
- In Section 4, we define the robust entropy-based risks, as the robust risk measures corresponding to the entropy-based risks. We show that these risks can be computed using convex optimization methods.
- In Section 5, we compare the performance of the Robust CVaR-optimal and CVaR-optimal portfolios under various probability measures. The probability distribution models were constructed using historical data of 20 stocks from various sectors traded in the New York Stock Exchange (NYSE) and the National Association of Securities Dealers Automated Quotations (NASDAQ) for the period between 2005 and 2010.

2. Robust risk measures and its representation

It is important to know whether the risk measure dened through the application of the robust optimization paradigm to deal with probability distribution uncertainty follows certain principles of consistent decision making, as the ones required in the denition of convex and coherent risk measures. The used terminology and mathematical background can be found in the appendix.

First, we present the definition of convex risk measures, and the representation theorem for them (Föllmer and Schied, 2002; Shapiro et al., 2009).

Definition 2.1. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and consider the random variables classes $L^1(\Omega, \mathcal{F}, \mathbb{P})$. A mapping $\rho : L^1(\Omega, \mathcal{F}, P) \to \overline{\mathbb{R}}$ is called a convex risk measure if it satisfies the following properties:

- Monotonicity: If $X_1, X_2 \in L^1(\Omega, \mathcal{F}, P)$, and $X_1 \leqslant X_2$, $\mathbb{P}$-a.s., then, $\rho(X_1) \geqslant \rho(X_2)$.
- Translation invariance: If $X, A \in L^1(\Omega, \mathcal{F}, P)$, and $A = \alpha$, $\mathbb{P}$-a.s., $\alpha \in \mathbb{R}$, then, $\rho(X + A) = \rho(X) - \alpha$.
- Convexity: If $X_1, X_2 \in L^1(\Omega, \mathcal{F}, P), 0 \leqslant \lambda \leqslant 1$, then, $\rho(\lambda X_1 + (1-\lambda)X_2) \leqslant \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$.

Let us fix once forever a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We view the probability $\mathbb{P}$ as a reference probability measure. To allow for probability changes in our model, and in contrast with the standard approach, we do not assume that the probability that drives the random process of our problem is $\mathbb{P}$, but merely one that is only minimally related to $\mathbb{P}$. Specifically, with $\mathcal{P}^0$ be the set of all probability measures on $(\Omega, \mathcal{F})$, let

$$\mathcal{P} \equiv \left\{ P \in \mathcal{P}^0 \middle| P \ll \mathbb{P} \text{ and } \frac{dP}{d\mathbb{P}} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \right\}, \qquad (2.1)$$

where $P \ll \mathbb{P}$ means that P is absolutely continuous with respect to $\mathbb{P}$, that is, that every $\mathbb{P}$-negligible set is also P-negligible, and $dP/d\mathbb{P}$