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Optimal allocation of redundancies in series systems ☆

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ABSTRACT

It is of great interest for the problem of how to allocate redundancies in a system so as to optimize the system performance in reliability engineering and system security. In this paper, we consider the problems of optimal allocation of both active and standby redundancies in series systems in the sense of various stochastic orderings. For the case of allocating one redundancy to a series system with two exponential components, we establish two likelihood ratio order results for active redundancy case and standby redundancy case, respectively. We also discuss the case of allocating K active redundancies to a series system and establish some new results. The results developed here strengthen and generalize some of the existing results in the literature. Specifically, we give an answer to an open problem mentioned in Hu and Wang [T. Hu, Y. Wang, Optimal allocation of active redundancies in r-out-of-n systems, Journal of Statistical Planning and Inference 139 (2009) 3733–3737]. Numerical examples are provided to illustrate the theoretic results established here.

1. Introduction

It is of great interest to allocate redundant component(s) in a system in order to optimize the lifetime of the resulting system in reliability engineering, and system security. This topic has posed many interesting theoretical problems to which many researchers have devoted themselves in the past decades; see, for example, Boland et al. (1992), Shaked and Shanthikumar (1992), Singh and Misra (1994, 1997), Valdés and Zequeira (2003, 2004, 2006), Valdés et al. (2010), Brito et al. (2011), Hu and Wang (2009), da Costa Bueno (2005), da Costa Bueno and do Carmo (2007), Misra et al. (2009, 2011a,b), and Li and Ding (2010) and the references therein.

In general, there are two ways to allocate redundancies to a system: active (or parallel) redundancy allocation, and standby redundancy allocation. The former is used when replacement of components during the operation of the system is impossible; in this case the redundancies are put in parallel to components of the systems which leads to taking the maximum of random variables. The latter is used when replacement of components during the operation of the system is possible; in this case the redundancy starts functioning immediately after the corresponding original component in the system fails which leads to taking the convolution of random variables.

Let X_1, X_2 and X be independent random variables representing the lifetimes of the components C_1, C_2 and the redundancy R, respectively. Suppose that S is a series system consisting of the components C_1 and C_2 and the problem is how to allocate the redundancy R so that the resulting system performs better. In active redundancy case, one wants to compare the lifetimes

$$U_1 = \wedge(\vee(X_1, X), X_2) \quad \text{and} \quad U_2 = \wedge(X_1, \vee(X_2, X)),$$

where the symbols '$\wedge$' and '$\vee$' mean min and max, respectively. In standby redundancy case, one wants to compare the lifetimes

$$W_1 = \wedge(X_1 + X, X_2) \quad \text{and} \quad W_2 = \wedge(X_1, X_2 + X).$$

Boland et al. (1992) proved that

$$U_1 \geqslant_{\text{st}} U_2 \iff X_1 \leqslant_{\text{st}} X_2,$$

where $\geqslant_{\text{st}}$ denotes the usual stochastic order and the formal definitions of various stochastic orders used in this paper will be given in Section 2. Singh and Misra (1994) showed if X_1, X_2 and X have exponential distributions with parameters λ_1, λ_2 and λ, then

$$\lambda_1 \geqslant \max\{\lambda_2, \lambda\} \Rightarrow U_1 \geqslant_{\text{hr}} U_2, \tag{1.1}$$

where $\geqslant_{\text{hr}}$ denotes the hazard rate order. For the standby redundancy case, Boland et al. (1992) showed that

$$X_1 \leqslant_{\text{hr}} X_2 \Rightarrow W_1 \geqslant_{\text{st}} W_2. \tag{1.2}$$

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Shaked and Shanthikumar (1992) first considered the problem of allocating K redundancies to a series system wherein the original components and redundancies have independent and identically distributed lifetimes. Let $\mathbf{K} = (k_1, \ldots, k_n)$ be an allocation vector with $\sum_{i=1}^n k_i = K$, i.e., k_i redundant components are put in parallel to the ith original component in the system. Let $T_s(\mathbf{K})$ denote the lifetime of the resulting series system. Shaked and Shanthikumar (1992) then established that

$$T_s(\mathbf{K}) \geqslant_{\rm st} T_s(\mathbf{K}') \quad \text{whenever } \mathbf{K}' \overset{\rm m}{\succeq} \mathbf{K}, \tag{1.3}$$

where $\overset{\rm m}{\succeq}$ denotes majorization order and the formal definitions on majorization type orders will be provided in Section 2. Singh and Singh (1997) further improved the result in (1.3) from the usual stochastic order to the hazard rate order as

$$T_s(\mathbf{K}) \geqslant_{\rm hr} T_s(\mathbf{K}') \quad \text{whenever } \mathbf{K}' \overset{\rm m}{\succeq} \mathbf{K}. \tag{1.4}$$

Hu and Wang (2009) and Misra et al. (2009) independently proved that, for a series system with two nodes,

$$T_s(k_1, k_2) \geqslant_{\rm rh} T_s(k'_1, k'_2) \quad \text{whenever } (k'_1, k'_2) \overset{\rm m}{\succeq} (k_1, k_2). \tag{1.5}$$

Hu and Wang (2009) also used a counterexample to show that (1.5) does not hold in general for the case when $n > 2$. However, they left an open problem whether the result in (1.5) may be strengthened to the likelihood ratio order.

The purposes of the present paper are twofold. The first purpose is to extend the comparison result for the problem of allocating one redundancy to the series system with two nodes under exponential framework. Specifically, it is shown if X_1, X_2 and X have exponential distributions with parameters λ_1, λ_2 and λ, respectively, then

$$\lambda_1 \geqslant \max\{\lambda_2, \lambda\} \Rightarrow U_1 \geqslant_{\rm lr} U_2, \tag{1.6}$$

where $\geqslant_{\rm lr}$ denotes the likelihood ratio order. Apparently, the result in (1.6) strengthens the corresponding one in (1.1) from the hazard rate order to the likelihood ratio order, and for the standby case, we show that

$$\lambda_1 \geqslant \lambda_2 \Rightarrow W_1 \geqslant_{\rm lr} W_2.$$

These two results will be proved in Section 3.

Another problem we focus on is to allocate K active redundancies to a series system which is treated in Section 4. For the case when the series system has two nodes, it is shown that

$$T_s(k_1, k_2) \geqslant_{\rm lr} T_s(k'_1, k'_2) \quad \text{whenever } (k'_1, k'_2) \overset{\rm m}{\succeq} (k_1, k_2), \tag{1.7}$$

which actually solves an open problem suggested by Hu and Wang (2009). In fact, we can reach a more general result than that in (1.7)

$$T_s(k_1, k_2) \geqslant_{\rm lr} T_s(k'_1, k'_2) \quad \text{whenever } (k'_1, k'_2) \overset{\rm w}{\succeq} (k_1, k_2).$$

We also establish that

$$T_s(k_1, k_2) \geqslant_{\rm rh} T_s(k'_1, k'_2)$$
$$\text{whenever } (k'_1 + 1, k'_2 + 1) \overset{\rm p}{\succeq} (k_1 + 1, k_2 + 1),$$

which allows the reliability engineer to obtain more reliable resulting system even though he/she has less redundancies. For the n-components series system, it is shown if $K = nk = \sum_{i=1}^n k_i$ and $\mathbf{K}_0 = (k, \ldots, k)$, then,

$$T_s(\mathbf{K}_0) \geqslant_{\rm lr} T_s(\mathbf{K}),$$

and if $(k^* + 1)^n = \prod_{i=1}^n (k_i + 1)$ and $\mathbf{K}^* = (k^*, \ldots, k^*)$, then,

$$T_s(\mathbf{K}^*) \geqslant_{\rm rh} T_s(\mathbf{K}). \tag{1.8}$$

We also show by a counterexample that the reversed hazard rate order in (1.8) cannot be replaced by the hazard rate order. Finally, it is shown that

$$T_s(\mathbf{K}) \geqslant_{\rm st} T_s(\mathbf{K}')$$
$$\text{whenever } (k'_1 + 1, \ldots, k'_{n+1}) \overset{\rm p}{\succeq} (k_1 + 1, \ldots, k_n + 1).$$

A counterexample is provided to illustrate that the usual stochastic order cannot be strengthened to the hazard rate order or the reversed hazard rate order.

2. Definitions and notations

In this section, we first recall some notions of stochastic orders, and majorization and related orders. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*.

2.1. Stochastic orders

Definition 2.1. For two random variables X and Y with densities f_X and f_Y, and distribution functions F_X and F_Y, respectively, let $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ be the corresponding survival functions. Denote by $h_X = f_X/\overline{F}_X[h_Y = f_Y/\overline{F}_Y]$ the hazard rate function of $X[Y]$, and $r_X = f_X/F_X[r_Y = f_Y/F_Y]$ the reversed hazard rate function of $X[Y]$. Then:

(i) X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leqslant_{\rm lr} Y$) if $f_Y(x)/f_X(x)$ is increasing in x.
(ii) X is said to be smaller than Y in the hazard rate order (denoted by $X \leqslant_{\rm hr} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in x; or equivalently, if $h_X(x) \geqslant h_Y(x)$.
(iii) X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \leqslant_{\rm rh} Y$) if $F_Y(x)/F_X(x)$ is increasing in x; or equivalently, if $r_X(x) \leqslant r_Y(x)$.
(iv) X is said to be smaller than Y in the usual stochastic order (denoted by $X \leqslant_{\rm st} Y$) if $\overline{F}_Y(x) \geqslant \overline{F}_X(x)$.

From Shaked and Shanthikumar (2007), the likelihood ratio order implies both the hazard rate order and the reversed hazard rate order which in turn imply the usual stochastic order, but neither the hazard rate order nor the reversed hazard rate order implies the other.

2.2. Majorization and related orders

The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 2.2. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$, written as $\mathbf{x} \overset{\rm m}{\succeq} \mathbf{y}$, if

$$\sum_{i=1}^j x_{(i)} \leqslant \sum_{i=1}^j y_{(i)} \quad \text{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$.

The majorization relation $\mathbf{x} \overset{\rm m}{\succeq} \mathbf{y}$ means the components of $\mathbf{x}$ are more equal than those of $\mathbf{y}$ (cf. Marshall and Olkin, 1979). In addition, the vector $\mathbf{x}$ is said to submajorize the vector $\mathbf{y}$ weakly, written as $\mathbf{x} \overset{\rm w}{\succeq} \mathbf{y}$, if

$$\sum_{i=1}^j x_{(i)} \leqslant \sum_{i=1}^j y_{(i)} \quad \text{for } j = 1, \ldots, n.$$