European Journal of Operational Research 220 (2012) 320–327

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Discrete Optimization

Some heuristic methods for solving *p*-median problems with a coverage constraint

Jesús Sáez-Aguado *, Paula Camelia Trandafir

Departamento de Estadística e Investigación Operativa Facultad de Ciencias, Universidad de Valladolid, c/Prado de la Magdalena s/n, 47005 Valladolid, Spain

ARTICLE INFO

Article history:
Received 25 October 2011
Accepted 7 February 2012
Available online 21 February 2012

Keywords:
Location
p-Median
Coverage constraint
Local search
GRASP
Lagrangean relaxation

ABSTRACT

The aim of this paper is to solve *p*-median problems with an additional coverage constraint. These problems arise in location applications, when the trade-off between distance and coverage is being calculated. Three kinds of heuristic algorithms are developed. First, local search procedures are designed both for constructing and improving feasible solutions. Second, a multistart GRASP heuristic is developed, based on the previous local search methods. Third, by employing Lagrangean relaxation methods, a very efficient Lagrangean heuristic algorithm is designed, which extends the well known algorithm of Handler and Zang, for constrained shortest path problems, to constrained *p*-median problems. Finally, a comparison of the computational efficiency of the developed methods is made between a variety of problems of different sizes.

1. Introduction

The *p*-median problem (PMP) is one of the most studied issues in combinatorial optimization, having many applications, among which cluster analysis [16,21], various location problems [10] and optimal diversity management [6] can be mentioned. For a general discussion of the problem, the reader is directed to [10,19].

Since Kariv and Hakimi [18] proved that the PMP is NP-hard, several heuristic algorithms have been found to solve the problem in an approximate way. Mladenovic et al. [20], Reese [24] and Alba and Domínguez [3] survey different approximations and metaheuristics designed for this problem, most of which are based on interchange or swap-based local search.

The interchange heuristic for the PMP, beginning with the well known Teitz and Bart algorithm [32], was improved with Whitaker's fast implementation [33] and, more recently, Resende and Werneck's [27,30]. Based on these implementations, Resende and Werneck [28,29] applied the GRASP (Greedy Randomized Adaptive Search Procedure) methodology to PMP, resulting in different algorithms which can be considered among the most efficient for the problem.

In this paper, the *p*-median problem with an additional coverage constraint (CONPMP) is considered. The constraint establishes that the total demand covered at a distance greater than some pre-established coverage distance should not exceed a previously chosen value. This constraint is related to the maximum covering location problem (MCLP), which is another important location problem (see [10,19]), whose objective is equivalent to minimizing the demand covered at a distance greater than the coverage distance. While the solution of the PMP is associated to maximum efficiency when designing a service, the solution of the CONPMP problem looks for maximum efficiency subject to a required minimum coverage.

CONPMP is a problem of interest in its own right, but it can also be used for exploring the trade-off between distance and coverage, by systematically varying the demand allowed to be covered at a distance greater than the coverage distance. This can be done, for example, by applying the ε-constraint method to the biobjective problem with extremum problems PMP and MCLP. This biobjective problem is considered in [10], which presents an elementary method, based on weighted sums, for finding a set of supported efficient solutions, and suggests that, with the ε-constraint method, all efficient solutions could be found. For an overview on available methods for solving multiobjective integer linear programming problems, the reader is directed to the excellent textbooks [9,11].

The main purpose of this paper is to provide methods for effectively solving the CONPMP subproblems. Although the aim is not to carry out a complete multiobjective analysis, there will be some discussion on the inherent trade-off between distance and coverage, which are the objectives of the PMP and MCLP problems, respectively. In a similar context, fixed charge facility location problems with coverage constrictions, and the corresponding trade-off between cost and coverage, have been considered in [23,34].

It is well known that adding a constraint to an optimization problem can make it much more difficult. This happens, for example, in the shortest-path [15], minimum spanning tree [2] or

* Corresponding author. Tel.: +34 983186417; fax: +34 983423013.
E-mail addresses: jsaez@eio.uva.es (J. Sáez-Aguado), camelia@eio.uva.es (P.C. Trandafir).

0377-2217/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.ejor.2012.02.011

assignment problems [1]. In these cases, an additional constraint turns an easy problem into an NP-hard one. In our case, PMP is already NP-hard, so adding the CONPMP constraint does not modify this complexity, but the problem is computationally more difficult.

In this paper, different heuristic procedures for the CONPMP problem are developed. The first group of algorithms are modified versions of the local search and interchange algorithms used in the PMP. In this way, local search procedures are designed both for constructing and improving feasible solutions for CONPMP. These procedures are the basis for the design of a multistart GRASP heuristic for the CONPMP problem. Given that GRASP procedures are among the most efficient for solving PMP, it is hoped that this will also be the case for CONPMP.

The next heuristic procedure for CONPMP is based on Lagrangean relaxation. Lagrangean relaxation is a well-known methodology for solving large-scale combinatorial optimization problems. It is based on exploiting the inherent structure of each problem in order to obtain lower and upper bounds on the optimal value of the problem. For surveys on the theory and applications of Lagrangean relaxation, see for instance [14,22]. Daskin [10] is an excellent reference on the application of Lagrangean relaxation techniques to different location problems.

Lagrangean relaxation based heuristics have been proposed in the literature for numerous combinatorial optimization problems. Some of them are: [10,5] for the PMP, [7] for the set covering problem, [4] for the capacitated facility location problem, [6] for the diversity management problem, [21] for the cluster analysis problem, [8] for the train timetabling problem and [31] for the fixed charge transportation problem.

Lagrangean relaxation techniques have very often been applied to problems with an additional constraint. So, in [15] Handler and Zang develop specific procedures for constrained shortest-path problems, based on Lagrangean relaxation. Similar procedures have been developed, for instance, for constrained assignment problems [1] and constrained minimum spanning tree [2], to mention only a few. On the other hand, in [17], the Handler and Zang algorithm is extended to general resource constrained optimization problems, and the running time of the resulting algorithm is established.

In this paper, we apply Lagrangean relaxation to the CONPMP problem based on the resolution of subproblems of type PMP. Instead of using classical procedures, such as subgradient optimization [21,22] or bisection search for the case of a unique multiplier [13], the extension of the well-known algorithm of Handler and Zang [15] for the restricted shortest path problem to CONPMP has been found to be much more efficient here. The extension of the Handler and Zang algorithm to other structured problems with an additional constraint is termed the Lagrangean Relaxation Based Aggregated Cost (LARAC) algorithm in [17,35]. The solution found by the LARAC method can, in addition, be improved with the local search heuristic, finally resulting in a very efficient heuristic method that provides solutions near optimality.

The paper is structured as follows. Section 2 states the problem and establishes the notation used in the rest of the article. Section 3 develops the basic procedures of local search for the CONPMP problem, which lead to the multistart GRASP algorithm in Section 4. Section 5 develops a Lagrangean relaxation which is the basis of the LARAC algorithm. Section 6 presents the computational results obtained when comparing the different approximations on two whole batteries of problems. Section 7 gives the final conclusions.

2. Statement and notation

CONPMP may be stated as follows:

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} h_i d_{ij} y_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_{ij} = 1 \quad i = 1,\dots,m \tag{2}$$

$$y_{ij} \leqslant x_j \quad i = 1,\dots,m,\ j = 1,\dots,n \tag{3}$$

$$\sum_{j=1}^{n} x_j = p \tag{4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} q_{ij} y_{ij} \leqslant \varepsilon \tag{5}$$

$$x_j \in \{0,\ 1\} \quad j = 1,\dots,n \tag{6}$$

$$y_{ij} \in \{0,\ 1\} \quad i = 1,\dots,m,\ j = 1,\dots,n \tag{7}$$

In this statement, m is the number of points of demand of a service, n the number of points of possible establishment of service, p the number of service points which will be opened, h_i the demand at point i, d_{ij} the distance between the demand point i and service point j; $x_j \in \{0, 1\}$ is the location variable, defined for $j = 1, \dots, n$, with $x_j = 1$ if a facility is located at point j, and $y_{ij} \in \{0, 1\}$ is the assignment variable, defined for $i = 1, \dots, m$, $j = 1, \dots, n$, with $y_{ij} = 1$ if all demand at point i is assigned to a facility at point j.

The objective function (1) expresses the sum total of distances traversed by all clients of the system. The constraints (2) indicate that all demand points must be serviced or assigned to a unique service point. The constraints (3) guarantee that if a service point is not open, then it cannot service any demand point, that is, $x_j = 0 \Rightarrow y_{ij} = 0, i = 1,\dots,m$. The constraints (4) mean that the total number of open installations must be the pre-established number p. Finally, (6) and (7) state that all variables take only the values 0 and 1.

The coefficients q_{ij} in the constraints (5) are defined as follows, based on a coverage distance DC established by the user:

$$q_{ij} = \begin{cases} h_i & \text{if} \quad d_{ij} > DC \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

So, the left hand side of (5) is the total demand covered at a distance greater than DC, which must not exceed ε, also specified by the user.

If we denote by $\mathcal{F}$ the feasible set given by the constraints (2)–(4) and (6) and (7), by $f_1(y) = \sum_{i=1}^{m}\sum_{j=1}^{n} h_i d_{ij} y_{ij}$, and by $f_2(y) = \sum_{i=1}^{m}\sum_{j=1}^{n} q_{ij} y_{ij}$, the problem can be stated in a more compact form as:

$$\text{(P)} \quad \text{Minimize} \quad f_1(y) \tag{9}$$
$$\text{subject to} \quad (x, y) \in \mathcal{F}$$
$$f_2(y) \leqslant \varepsilon \tag{10}$$

It should be noted that $\mathcal{F}$ is the feasible set of a PMP problem without constraints. To avoid trivial cases, in all that follows, it will be supposed that $\mathcal{F}$ is nonempty. It will also be assumed that exact or approximate algorithms are available to solve PMP problems with any linear objective.

Regarding the feasibility of problem (P), for $k = 1, 2$, consider first the two PMP problems:

$$(P_k) \quad \text{Minimize} \quad f_k(y)$$
$$\text{subject to} \quad (x, y) \in \mathcal{F} \tag{11}$$

Let $(x^k, y^k) \in \mathcal{F}$ be an optimal solution of (P_k). Defining $\varepsilon_1 = f_2(y^1)$ and $\varepsilon_2 = f_2(y^2)$, the following is immediately verified:

Proposition 1. *If the feasible set for PMP $\mathcal{F}$ is nonempty, then*

1. *Problem (P) is feasible if and only if $\varepsilon_2 \leqslant \varepsilon$.*
2. *If $\varepsilon \geqslant \varepsilon_1$, then (x^1, y^1) fulfills the constraint $f_2(y) \leqslant \varepsilon$, and is therefore an optimum for (P).*