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Continuous Optimization

On solving the planar k-centrum problem with Euclidean distances

Antonio M. Rodríguez-Chía [a,*], Inmaculada Espejo [a], Zvi Drezner [b]

[a] Department of Statistics and Operations Research, Universidad de Cádiz, Spain
[b] Steven G. Mihaylo College of Business and Economics, California State University-Fullerton, Fullerton, CA 92834, USA

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ABSTRACT

This paper presents a solution procedure based on a gradient descent method for the k-centrum problem in the plane. The particular framework of this problem for the Euclidean norm leads to bisector lines whose analytical expressions are easy to handle. This allows us to develop different solution procedures which are tested on different problems and compared with existing procedures in the literature of Location Analysis. The computational analysis reports that our procedures provide better results than the existing ones for the k-centrum problem.

1. Introduction

Location Analysis is one of the most active fields in Operations Research; in fact, many different models have been developed in the last few decades to deal with different real world applications. Most classical location studies focus on the minimization of the total distance (the median concept) or the minimization of the maximum distance (the center concept) to the service facilities (see Drezner and Hamacher, 2002; Drezner, 1995; Francis et al., 1992; Love et al., 1988). The median solution concept is primarily concerned with the spatial efficiency while the center concept is focused on the spatial equity. The k-centrum model unifies both concepts by minimization of the sum of the k largest distances (if $k = 1$ the model coincides with the standard center problem while $k = M$ defines the classical median problem).

The k-centrum concept was first defined by Slater (1978a) and Andreatta and Mason (1985a) for the discrete single facility location problem on a tree graph. The k-centrum concept was later extended to general location problems covering discrete as well as continuous decisions in Ogryczak and Zawadzki (2002). Recently, Ogryczak and Tamir (2003) reformulated the single-facility rectilinear k-centrum problem as a linear program through replacing each nonlinear constraint composed of a rectilinear distance function by a set of $2m$ linear constraints, and then gave a polynomial time algorithm with better complexity bound than those reported by Tamir (2001) and Kalcsics et al. (2002) for network location problems. An equivalent formulation to the one introduced by Ogryczak and Tamir (2003) is given in Ogryczak and Sliwinski (2003).

In this paper we consider the k-centrum problem with the Euclidean norm in the plane. The current research for Euclidean facility location problems mainly focuses on the median problem, for which many practical and efficient algorithms have been designed since Weiszfeld presented a simple iterative algorithm in 1937 (translated by Plastria in Weiszfeld and Plastria (2009)). These include the hyperbolic approximation procedure (Morris and Verdini, 1979; Frenk et al., 1994), the interior point algorithms (Andersen et al., 2000; Xue et al., 1996) and the smoothing Newton methods (Qi et al., 2002). However, the solution methods for the k-centrum location problem are rarely seen in the literature. In Alizadeh and Goldfarb (2003), the problem of minimizing the largest Euclidean norms is only mentioned as a special example of a second-order cone programming and consequently can be solved by an interior point method. In Pan and Li (2005) and Pan and Chen (2007), they reformulate the single-facility Euclidean k-centrum location problem in $\mathbb{R}^n$ as a nonsmooth optimization problem and develop a smoothing algorithm. Also, in the Euclidean case, Lozano et al. (2010a) uses the k-centrum criterium to locate a straight line and Lozano et al. (2010b) considers the obnoxious version of this criterium, i.e., maximizing the sum of the k-smallest distances. These both references consider the weighted and unweighted versions.

The k-centrum problem is a particular case of the ordered median problem, which unifies many classical and new facility location problems (see Puerto and Fernández, 1995; Nickel and Puerto, 2005). This type of objective function, which generalizes the most popular

* Corresponding author.
E-mail address: antonio.rodriguezchia@uca.es (A.M. Rodríguez-Chía).

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objective functions, has attracted the attention of researches in the last years. In network location problems, efforts have been devoted to obtain finite dominating sets and efficient algorithms to solve this kind of problems (Kalcsics et al., 2003; Nickel and Puerto, 1999; Puerto and Rodríguez-Chía, 2005). The discrete versions of these models have also been studied in Boland et al. (1993), Domínguez-Marín et al. (2005), Kalcsics et al. (2009a, in press); Marin et al. (2009) and Nickel (2000). For the planar case with polyhedral gauges, Rodríguez-Chía et al. (2000) develop a polynomial time algorithm and Puerto et al. (1997) and Saameño et al. (2006) apply these models to semiobnoxious location problems. Ohsawa et al. (2007) consider the unweighted ordered median problem with squared Euclidean distances and, taking advantage of the fact that in this case the level curves are circumferences, they give a characterization of the solution set. The multicriteria case is also studied in Nickel et al. (2005). Espejo et al. (2009) present a procedure to solve the convex ordered median problem where the distances are measured with ℓ_p-norms. They develop an algorithm based on a gradient descent method that generates a sequence with decreasing objective value. Drezner and Nickel (2009a) propose a general approach solution method, based on the big triangle small triangle method, for the single facility ordered median problem in the plane with the Euclidean metric and equal weights associated with each demand point. Recently, Drezner and Nickel (2009b) show that every ordered median function can be expressed as a difference of two convex functions (DC) and apply general approaches for DC optimization to solve the ordered one median problem in the plane.

In this paper, we propose an iterative procedure for solving the k-centrum problem inspired by the modified gradient descent method given in Espejo et al. (2009). This methodology is complex because the objective function does not have a common expression as sum of the weighted distances from the clients to the server; in fact, it is pointwise defined. We exploit the properties of the k-centrum problem with the Euclidean norm to develop two solution methods for the cases of equal and different weights associated with the demand points.

The paper is organized as follows. In the next section we introduce the Euclidean convex ordered median problem. In Section 3, the procedure for solving this problem is presented. The particular case of equal weights associated with the demand points is studied in Section 4. We will adapt the procedures for solving the k-centrum problem in Section 5. Finally, all the proposed procedures are tested on different problems and compared with existing algorithms in the literature for solving the k-centrum problem. The paper ends with some conclusions and an outlook to future research.

2. The convex ordered median problem

In this section we present the convex ordered median problem with the Euclidean metric in the plane. Consider the set of demand points in the plane $\mathcal{A} = \{a_1, \dots, a_M\}$ and two vectors of nonnegative scalars $\omega := (\omega_1, \dots, \omega_M)$ and $\lambda := (\lambda_1, \dots, \lambda_M)$, where $\lambda_1 \leqslant \dots \leqslant \lambda_M$. The component ω_i is the weight corresponding to the importance given to the existing facility a_i, $i = 1, \dots, M$ and the components of the λ-vector give us the possibility of choosing among different kinds of objective functions. Observe that the ω-vector allows us to take into account factors as the population of a city or the intensity of the demand (see Kalcsics et al., 2002, 2003; Nickel and Puerto, 2005; Ogryczak and Tamir, 2003; Rodríguez-Chía et al., 2000; Puerto and Rodríguez-Chía, 2005, among others).

Given a permutation σ of $\{1, \dots, M\}$ verifying

$$\omega_{\sigma_1}\|x - a_{\sigma_1}\|_2 \leqslant \dots \leqslant \omega_{\sigma_M}\|x - a_{\sigma_M}\|_2,$$

where $\|\cdot\|_2$ denotes the ℓ_2 norm, we define

$$d_{(i)}(x) := \omega_{\sigma_i}\|x - a_{\sigma_i}\|_2.$$

The ordered median problem is given by the following formulation:

$$\min_{x \in \mathbb{R}^2} F(x) = \sum_{i=1}^{M} \lambda_i d_{(i)}(x). \tag{1}$$

For different choices of λ we obtain different types of objective functions including the classical location problems as median problems ($\lambda = (1, 1, \dots, 1)$), center problems ($\lambda = (0, 0, \dots, 0, 1)$), μ-centdian problems ($\lambda = (\mu, \mu, \dots, \mu, 1)$ for $0 < \mu < 1$) or k-centrum problems ($\lambda = (0, \dots, 0, \overbrace{1, \dots, 1}^{k})$), among others.

We can also see that this objective function is pointwise defined. This means that the objective function has different explicit expressions as sum of the weighted distances to the demand points, depending on the order of the sequence of the weighted distances. Indeed, for instance taking $a_1 = (0, 0)$ and $a_2 = (0, 5)$, with $\omega_1 = \omega_2 = 1$ and $\lambda = (\lambda_1, \lambda_2)$, the ordered median function at $x = (2, 1)$ and $y = (2, 4)$ is given by (see Fig. 1)

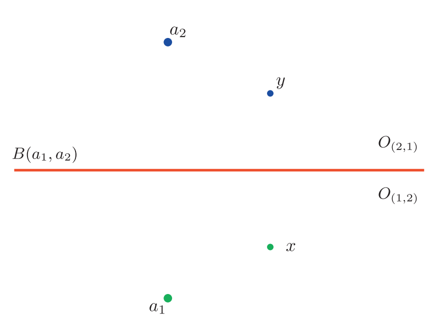

Fig. 1. $B(a_1, a_2)$ for $\omega_1 = \omega_2 = 1$.