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Original Article

## Computation of Smarandache curves according to Darboux frame in Minkowski 3-space

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## ABSTRACT

In this paper, we study Smarandache curves according to Darboux frame in the three-dimensional Minkowski space  $E_1^3$ . Using the usual transformation between Frenet and Darboux frames, we investigate some special Smarandache curves for a given timelike curve lying fully on a timelike surface. Finally, we defray a computational example to confirm our main results.

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## 1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are very interesting and important problem.

Among all space curves, Smarandache curves have special emplacements regarding their properties, because of this they deserve especial attention in Euclidean geometry as well as in other geometries. It is known that Smarandache geometry is a geometry which has at least one Smarandache denied axiom [1]. An axiom is said to be Smarandache denied, if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes. Smarandache curves are the objects of Smarandache geometry. By definition, if the position vector of a curve  $\delta$  is composed by Frenet frame's vectors of another curve  $\beta$ , then the curve  $\delta$  is called a Smarandache curve [2]. In three-dimensional curve theory, each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields  $\mathbf{T}$ ,  $\mathbf{N}$  and  $\mathbf{B}$

(Frenet frame vectors) are respectively, the tangent, the principal normal and the binormal vector fields.

In the light of the existing studies in the field of geometry, many interesting results on Smarandache curves have been obtained by many mathematicians [3–9]. Turgut and Yilmaz have introduced a particular circumstance of such curves, they entitled it Smarandache  $\mathbf{TB}_2$  curves in the space  $E_4^1$  [2]. They studied special Smarandache curves which are defined by the tangent and second binormal vector fields. In [4], the author has illustrated certain special Smarandache curves in the Euclidean space.

Special Smarandache curves in such a manner that Smarandache curves  $\mathbf{TN}_1$ ,  $\mathbf{TN}_2$ ,  $\mathbf{N}_1\mathbf{N}_2$  and  $\mathbf{TN}_1\mathbf{N}_2$  with respect to Bishop frame in Euclidean 3-space have been sought for by Çetin et al. [6]. Furthermore, they worked differential geometric properties of these special curves and checked out first and second curvatures of these curves. Also, they get the centers of the curvature spheres and osculating spheres of Smarandache curves.

Recently, H.S. Abdel-Aziz and M. Khalifa Saad have studied special Smarandache curves of an arbitrary curve such as  $\mathbf{TN}$ ,  $\mathbf{TB}$  and  $\mathbf{TNB}$  with respect to Frenet frame in the three-dimensional Galilean and pseudo-Galilean spaces [3,7].

The main goal of this article is to introduce and describe some special Smarandache curves in  $E_1^3$  for a given timelike surface and a timelike curve lying fully on it with reference to its Darboux

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frame. We looking forward to that our results will be helpful to researchers who are specialized on mathematical modeling and other applications.

2. Basic concepts

Let us first recall the basic notions from Lorentz geometry [10].

Let  $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$  be a 3-dimensional vector space, and let  $\mathbf{x} = (x_1, x_2, x_3)$  and  $\mathbf{y} = (y_1, y_2, y_3)$  be two vectors in  $\mathbb{R}^3$ . The Lorentz scalar product of  $\mathbf{x}$  and  $\mathbf{y}$  is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle_L = x_1y_1 + x_2y_2 - x_3y_3.$$

$E_1^3 = (\mathbb{R}^3, \langle \mathbf{x}, \mathbf{y} \rangle_L)$  is called 3-dimensional Lorentzian space, Minkowski 3-space or 3-dimensional Semi-Euclidean space. The arbitrary vector  $\mathbf{x}$  in  $E_1^3$  can have one of three Lorentzian causal characters; it can be spacelike, timelike and lightlike (null) if  $\langle \mathbf{x}, \mathbf{x} \rangle_L > 0$  or  $\mathbf{x} = 0$ ,  $\langle \mathbf{x}, \mathbf{x} \rangle_L < 0$  and  $\langle \mathbf{x}, \mathbf{x} \rangle_L = 0$  and  $\mathbf{x} \neq 0$ , respectively. Similarly, a curve  $r$ , locally parameterized by  $r = r(s) : I \subset \mathbb{R} \rightarrow E_1^3$  where  $s$  is pseudo arclength parameter, is called a spacelike curve if  $\langle r'(s), r'(s) \rangle_L > 0$ , timelike if  $\langle r'(s), r'(s) \rangle_L < 0$  and lightlike if  $\langle r'(s), r'(s) \rangle_L = 0$  and  $r'(s) \neq 0$  for all  $s \in I$ . The two vectors  $\mathbf{x} = (x_1, x_2, x_3)$ ,  $\mathbf{y} = (y_1, y_2, y_3) \in E_1^3$  are orthogonal if and only if  $\langle \mathbf{x}, \mathbf{y} \rangle_L = 0$ . Also, for any  $\mathbf{x}, \mathbf{y} \in E_1^3$ , Lorentzian vector product of  $\mathbf{x}$  and  $\mathbf{y}$  is defined by

$$\mathbf{x} \times_L \mathbf{y} = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

The norm of a vector  $\mathbf{x} \in E_1^3$  is given by  $\|\mathbf{x}\|_L = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle_L|}$ . We denote by  $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$  the moving Frenet frame along the curve  $r(s)$  in the Minkowski space  $E_1^3$ , where the vectors  $\mathbf{T}, \mathbf{N}, \mathbf{B}$  are called the vectors of the tangent, principal normal and the binormal of  $r$ , respectively. Consider now the following definition that we are needed throughout this study.

**Definition 1.** A surface  $M$  in the Minkowski 3-space  $E_1^3$  is said to be spacelike, timelike surface if, respectively the induced metric on the surface is a positive definite Riemannian metric, Lorentz metric. In other words, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [10].

3. Smarandache curves of a timelike curve on a timelike surface

Consider a timelike curve  $r = r(s)$  in  $E_1^3$ , parameterized by its arc length  $s$  and lying fully on an oriented timelike surface  $M$ . Let  $\mathbf{T}, \mathbf{N}, \mathbf{B}$  be the tangent, principal normal and binormal vector fields along  $r(s)$ . Then, Frenet frame is given by

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{1}$$

where a prime denotes differentiation with respect to  $s$ . For this frame the following are satisfying

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= -1, \quad \langle \mathbf{B}, \mathbf{B} \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{B} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = 0. \end{aligned}$$

The coefficients  $\kappa$  and  $\tau$  are the curve's curvature and torsion.

Let  $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$  be the Darboux frame of  $r(s)$  with  $\mathbf{T}$  as the tangent vector of  $r$  and  $\mathbf{U}$  is the unit normal to the surface  $M$  and  $\mathbf{P} = \mathbf{T} \times \mathbf{U}$ , then the usual transformation between Frenet and Darboux frames takes the form [10,11]:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2}$$

where  $\theta$  is the angle between the vectors  $\mathbf{P}$  and  $\mathbf{N}$ . Therefore, the Darboux frame of  $r(s)$  is given as follows

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{P}' \\ \mathbf{U}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_N \\ \kappa_g & 0 & -\tau_g \\ \kappa_N & \tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{U} \end{pmatrix}, \tag{3}$$

where  $\kappa_N, \kappa_g$  and  $\tau_g$  are the normal curvature, geodesic curvature and geodesic torsion, respectively. In the differential geometry of surfaces, for a curve  $r = r(s)$  lying on a surface  $M$  the following are well-known [11]

- (i)  $r(s)$  is a geodesic curve if and only if  $\kappa_g = 0$ .
- (ii)  $r(s)$  is an asymptotic line if and only if  $\kappa_N = 0$ .
- (iii)  $r(s)$  is a principal line if and only if  $\tau_g = 0$ .

**Definition 2.** A regular curve in Minkowski 3-space, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [2].

In the following, we investigate Smarandache curves  $\mathbf{TP}, \mathbf{TU}, \mathbf{PU}$  and  $\mathbf{TPU}$  and study some of their properties which represent the main results.

3.1. TP–Smarandache curves

From the Definition 2, the TP–Smarandache curve of  $r$  can be defined by

$$\alpha(\bar{s}) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{P}). \tag{4}$$

Differentiating Eq. (4) with respect to  $s$ , we obtain

$$\alpha' = \frac{d\alpha}{d\bar{s}} \frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}(\kappa_g \mathbf{T} + \kappa_g \mathbf{P} + (\kappa_N - \tau_g) \mathbf{U}),$$

with the parameterization

$$\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}(\kappa_N - \tau_g), \tag{5}$$

and then

$$\bar{\mathbf{T}}_\alpha = \frac{1}{(\kappa_N - \tau_g)}(\kappa_g \mathbf{T} + \kappa_g \mathbf{P} + (\kappa_N - \tau_g) \mathbf{U}). \tag{6}$$

Differentiating Eq. (6) with respect to  $s$  and using Eq. (5), we get

$$\frac{d\bar{\mathbf{T}}_\alpha}{d\bar{s}} = \frac{-\sqrt{2}}{(\kappa_N - \tau_g)^3}(\omega_1 \mathbf{T} + \omega_2 \mathbf{P} + \omega_3 \mathbf{U}),$$

where

$$\omega_1 = (\tau_g' \kappa_g - \kappa_N' \kappa_g + (\kappa_N - \tau_g)(\kappa_g' + \kappa_g^2 + \kappa_N^2 - \kappa_N \tau_g)),$$

$$\omega_2 = (\tau_g' \kappa_g - \kappa_N' \kappa_g + (\kappa_N - \tau_g)(\kappa_g' + \kappa_g^2 + (\kappa_N - \tau_g)\tau_g)),$$

$$\omega_3 = \kappa_g(\kappa_N - \tau_g)^2.$$

The curvature and torsion of  $\alpha$  are given as follows

$$\bar{\kappa}_\alpha = \left\| \frac{d\bar{\mathbf{T}}_\alpha}{d\bar{s}} \right\| = \frac{1}{(\kappa_N - \tau_g)^3} \sqrt{2(-\omega_1^2 + \omega_2^2 - \omega_3^2)}, \tag{7}$$

and

$$\bar{\mathbf{N}}_\alpha = \frac{\omega_1 \mathbf{T} + \omega_2 \mathbf{P} + \omega_3 \mathbf{U}}{\sqrt{-\omega_1^2 + \omega_2^2 - \omega_3^2}}.$$

On the other hand, we express

$$\bar{\mathbf{B}}_\alpha = -(\bar{\mathbf{T}}_\alpha \times \bar{\mathbf{N}}_\alpha) = \frac{-1}{(\kappa_N - \tau_g) \sqrt{-\omega_1^2 + \omega_2^2 - \omega_3^2}}$$

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