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Fixed point theorems for a generalized contraction mapping of rational type in symmetric spaces

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ABSTRACT

In this work, we establish some fixed point results for a contraction of rational type in symmetric spaces extending the fixed point theorems of Almeida, Roldan-Lopez-de-Hierro and Sadarangani [1].

1. Introduction and preliminaries

It worth to mention that the use of triangle inequality in a metric space (X, d) is of extreme importance since it implies that (i) d is continuous, (ii) each open ball is an open set, (iii) a sequence may converge to a unique point, (iv) every convergent sequence is a Cauchy sequence and other things. One of the importance generalizations of metric spaces is symmetric spaces, where the triangle inequality is relaxed. It was not immediately observed that such spaces may fail to satisfy properties (i)–(iv). Hence, in some of last papers, the authors implicitly used some of conditions (i)–(iv), so that their results were inaccuracy. Various authors introduced many types, generalizations, and applications of generalized metric spaces until now (see, [2–5]).

On the other hand, In 2015, Almeida, Roldan-Lopez-de-Hierro and Sadarangani [1] proved that whenever f is a rational type contraction mapping from a complete metric space into itself, then f has a unique fixed point. In this paper, we introduce fixed point theorems for contraction mappings of rational type in symmetric spaces. Our results generalize the results due to Almeida, Roldan-Lopez-de-Hierro and Sadarangani [1].

Next, we present some preliminaries and notations related to symmetric spaces and rational type contractions.

Definition 1.1. [6]. Suppose that X be a non-empty set and S: $X \times X \to [0, \infty)$ be a distance function such that:

(i) $S(x, y) = 0 \Leftrightarrow x = y$.
(ii) $S(x, y) = S(y, x)$,

for all $x, y \in X$.

We mean by a pair (X, S) with a symmetric space.

Definition 1.2. [6]. Let (X, S) be a symmetric space.

(a) A sequence $\{x_n\}$ in X is S-Cauchy sequence if $\lim_{n\to\infty} S(x_n, x_{n+r}) = 0$, $r \in N$(the set of all natural numbers).
(b) (X, S) is S-complete if for every S-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} S(x_n, x) = 0$.
(c) f: $X \to X$ is S-continuous if $\lim_{n\to\infty} S(x_n, x) = 0$ implies $\lim_{n\to\infty} S(fx_n, fx) = 0$.

We need the following properties in a symmetric space (X, S).

(W_3) [7] Given $\{x_n\}$, y and x in X ,$\lim_{n\to\infty} S(x_n, x) = 0$ and $\lim_{n\to\infty} S(x_n, y) = 0$ imply that $x = y$.

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(W_4) [7] Given $\{x_n\}$, $\{y_n\}$ and x in X ,$\lim_{n\to\infty} S(x_n, x) = 0$ and $\lim_{n\to\infty} S(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} S(y_n, x) = 0$.
(1C) [8] A function S is 1-continuous if $\lim_{n\to\infty} S(x_n, x) = 0 \Longrightarrow \lim_{n\to\infty} S(x_n, y) = S(x, y)$.

Remark 1.1. [7]. $(W_4) \Longrightarrow (W_3)$.

Definition 1.3. [9]. Let f: $X \to X$ and β: $X \times X \to [0, \infty)$. The mapping f is $\beta-$admissible if, for all $x, y \in X$ such that $\beta(x, y) > 1$, we have $\beta(fx, fy) > 1$.

Definition 1.4. [9]. Let (X, S) be a symmetric space and β: $X \times X \to [0, \infty)$. X is $\beta-$regular if, for each sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}) > 1$ for all $n \in N$ and $\lim_{n\to\infty} x_n = x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\beta(x_{n_k}, x) > 1$ $\forall$ $k \in N$.

In 2011, Haghi et al. [10] showed that some coincidence point and common fixed point generalizations in fixed point theory are not real generalizations. They gave the following lemma which show that the authors should take care in obtaining real generalizations in fixed point theory.

Lemma 1.1. *[10]. Let X be a nonempty set and f: X → X a function. Then there exists a subset E⊆X such that f(E) = f(X) and f: E → X is one-to-one.*

2. Main results

In this section we introduce some new fixed point results for a rational contraction self-mapping on symmetric spaces.

Theorem 2.1. *Suppose that* (X, S) *be a S-complete symmetric space satisfy* (W_4) *and* $(1C)$. *Let f be a self-mapping on X, and the following condition holds:*

$$S(fx, fy) \le \phi(M(x, y)) + C \min\{S(x, fx), S(y, fy), S(x, fy), S(y, fx)\} \; \forall \, x, y \in X, \; C \ge 0, \quad (1)$$

where $M(x, y)$ *is defined by*

$$M(x, y) = \max\left\{S(x, y), \frac{S(x, fx)(S(y, fy) + 1)}{1 + S(x, y)}, \frac{S(y, fy)(S(x, fx) + 1)}{1 + S(x, y)}\right\}.$$

and ϕ: $[0, \infty) \to [0, \infty)$ *be a continuous, nondecreasing function and* $\lim_{n\to\infty} \phi^n(t) = 0 \; \forall \, t > 0$.

Then f have a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be the sequence defined by $x_{n+1} = fx_n$ for all $n \in N$. If there exists $m \in N$ such that $x_m = x_{m+1}$, then $x_m = x_{m+1} = fx_m$, so x_m is a fixed point of f. In this case, the proof is finished. Suppose, on the contrary, that $x_{n+1} \neq x_n$ for all $n \in N$, that is $d(x_n, x_{n+1}) > 0$.

By (1), we have

$$\begin{aligned} S(fx_n, fx_{n+1}) &\le \phi(M(x_n, x_{n+1})) \\ &\quad + C \min\{S(x_n, fx_n), S(x_{n+1}, fx_{n+1}), \\ &\quad S(x_n, fx_{n+1}), S(x_{n+1}, fx_n)\} \\ &= \phi(M(x_n, x_{n+1})) \end{aligned} \quad (2)$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{S(x_n, x_{n+1}), \frac{S(x_n, fx_n)(S(x_{n+1}, fx_{n+1}) + 1)}{1 + S(x_n, x_{n+1})}, \frac{S(x_{n+1}, fx_{n+1})(S(x_n, fx_n) + 1)}{1 + S(x_n, x_{n+1})}\right\} \\ &= \max\left\{S(x_n, x_{n+1}), \frac{S(x_n, x_{n+1})(1 + S(x_{n+2}, x_{n+1}))}{1 + S(x_n, x_{n+1})}, S(x_{n+2}, x_{n+1})\right\}, \end{aligned}$$

we consider the following cases

- If $M(x_n, x_{n+1}) = S(x_n, x_{n+1})$ from (2) we have
$$S(x_{n+1}, x_{n+2}) \le \phi(S(x_n, x_{n+1})) < S(x_n, x_{n+1}) \quad (3)$$
- If $M(x_n, x_{n+1}) = \frac{S(x_n, x_{n+1})(1+S(x_{n+2}, x_{n+1}))}{1+S(x_n, x_{n+1})}$ from (2) we obtain
$$\begin{aligned} S(x_{n+1}, x_{n+2}) &\le \phi\left(\frac{S(x_n, x_{n+1})(1 + S(x_{n+2}, x_{n+1}))}{1 + S(x_n, x_{n+1})}\right) \\ &< \frac{S(x_n, x_{n+1})(1 + S(x_{n+2}, x_{n+1}))}{1 + S(x_n, x_{n+1})}. \end{aligned}$$
Hence
$$S(x_{n+1}, x_{n+2}) < S(x_n, x_{n+1}),$$
that is (3) holds.
- If $M(x_n, x_{n+1}) = S(x_{n+2}, x_{n+1})$ from (2) we get
$$S(x_{n+2}, x_{n+1}) < S(x_{n+2}, x_{n+1}),$$

which is impossible.

In any case, we proved that (3) holds. Since $\{S(x_{n+1}, x_{n+2})\}$ is decreasing. Hence, it converges to a nonnegative number, $c \ge 0$. If $c > 0$, then letting $n \to +\infty$ in (2), we deduce

$$c \le \phi\left(\max\left\{c, \frac{c(1 + c)}{1 + c}, c\right\}\right) = \phi(c) < c,$$

which implies that $c = 0$, that is

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+2}) = 0. \quad (4)$$

By using (W_4) and for any integer number r we have

$$\lim_{n\to\infty} S(x_n, x_{n+r}) = 0, \quad (5)$$

which implies that $\{x_n\}$ is S-Cauchy sequence. Since (X, S) is S-complete, there exists $u \in X$ such that $\lim_{n\to\infty} S(x_n, u) = 0$. From (W_4) we have

$$\lim_{n\to\infty} S(x_{n+1}, u) = 0.$$

Let $u \neq fu$. Applying (1) and using (1C) we get

$$\begin{aligned} S(fu, u) &= \lim_{n\to\infty} S(fu, x_{n+1}) = \lim_{n\to\infty} S(fu, fx_n) \\ &\le \lim_{n\to\infty} [\phi(M(u, x_n)) + C \min\{S(x_n, fx_n), \\ &\quad S(u, fu), S(x_n, fu), S(u, fx_n)\}] \\ &= \lim_{n\to\infty} [\phi(M(u, x_n)) + C \min\{S(x_n, x_{n+1}), \\ &\quad S(u, fu), S(x_n, fu), S(u, x_{n+1})\}] \\ &= \lim_{n\to\infty} [\phi(M(u, x_n))] < S(fu, u), \end{aligned} \quad (6)$$

where

$$\begin{aligned} M(u, x_n) &= \max\left\{S(u, x_n), \frac{S(u, fu)(S(x_n, fx_n) + 1)}{1 + S(u, x_n)}, \frac{S(x_n, fx_n)(S(u, fu) + 1)}{1 + S(u, x_n)}\right\} \\ &= \max\left\{S(u, x_n), \frac{S(u, fu)(S(x_n, x_{n+1}) + 1)}{1 + S(u, x_n)}, \right. \end{aligned}$$

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