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Original Article

Oscillation criteria for higher order quasilinear dynamic equations with Laplacians and a deviating argument

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ABSTRACT

In this paper, we deal with the oscillation of the solutions of the higher order quasilinear dynamic equation with Laplacians and a deviating argument in the form of

$$(x^{[n-1]})^{\Delta}(t)+p(t)\phi_{\gamma}(x(g(t)))=0$$

on an above-unbounded time scale, where $n\geq 2$,

$$x^{[i]}(t):=r_i(t)\phi_{\alpha_i}\left[\left(x^{[i-1]}\right)^{\Delta}(t)\right],\quad i=1,2,\ldots,n-1,\quad \text{with } x^{[0]}=x.$$

By using a generalized Riccati transformation and integral averaging technique, we establish some new oscillation criteria for the cases when n is even and odd, and when $\alpha>\gamma$, $\alpha=\gamma$, and $\alpha<\gamma$, respectively, with $\alpha=\alpha_1\cdots\alpha_{n-1}$ and without any restrictions on the time scale.

1. Introduction

In this paper we study the oscillatory behavior of the higher order quasilinear dynamic equation with Laplacians and a deviating argument

$$(x^{[n-1]})^{\Delta}(t)+p(t)\phi_{\gamma}(x(g(t)))=0 \tag{1.1}$$

on an above-unbounded time scale $\mathbb{T}$, where

(i) $n\geq 2$ is an integer and $\gamma>0$;
(ii) $x^{[i]}(t):=r_i(t)\phi_{\alpha_i}\left[\left(x^{[i-1]}\right)^{\Delta}(t)\right]$, $i=1,2,\ldots,n-1$, with $x^{[0]}=x$;
(iii) $\phi_{\theta}(u):=|u|^{\theta}\operatorname{sgn}u$ for $\theta>0$;

Without loss of generality we assume $t_0\in\mathbb{T}$. For $A\subset\mathbb{T}$ and $B\subset\mathbb{R}$, we denote by $C_{rd}(A, B)$ the space of right-dense continuous functions from A to B and by $C^1_{rd}(A,B)$ the set of functions in $C_{rd}(A, B)$ with right-dense continuous $\Delta-$derivatives, for an excellent introduction to the calculus on time scales, see Bohner and Peterson [1,2]. Throughout this paper we make the following assumptions:

(iv) For $i=1,2,\ldots,n-1$, $\alpha_i>0$ is a constant and $r_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ such that

$$\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(\tau)\Delta\tau=\infty; \tag{1.2}$$

(v) $p\in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,\infty))$ such that $p\not\equiv 0$;
(vi) $g\in C_{rd}(\mathbb{T},\mathbb{T})$ such that $\lim_{t\to\infty}g(t)=\infty$ with $g^*(t):=\min\{t, g(t)\}$ is nondecreasing on $[t_0,\infty)_{\mathbb{T}}$.

By a solution of Eq. (1.1) we mean a function $x\in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ for some $T_x\geq 0$ such that $x^{[i]}\in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, $i=1,2,\ldots,n-1$, which satisfies Eq. (1.1) on $[T_x,\infty)_{\mathbb{T}}$. A solution $x(t)$ of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is nonoscillatory.

Oscillation criteria for higher order dynamic equations on time scales have been studied by many authors. For instance, Grace, Agarwal, and Zafer [3] established oscillation criteria for the higher order nonlinear dynamic equations on general time scales

$$x^{\Delta^n}(t)+p(t)(x^{\sigma}(g(t)))^{\gamma}=0,$$

where γ is ratios of positive odd integers and where $g(t)\leq t$. In [3], some comparison criteria have been obtained when $g(t)\leq t$ and some oscillation criteria are given when n is even and $g(t)=t$.

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The authors in [3] assumed that

$$\int_{t_0}^{\infty}\int_t^{\infty}\int_s^{\infty} p(u)\Delta u\Delta s\Delta t = \infty. \tag{1.3}$$

Wu et al [4] established Kamanev-type oscillation criteria for the higher order nonlinear dynamic equation

$$\{r_{n-1}(t)[(r_{n-2}(t)(\ldots(r_1(t)x^{\Delta}(t))^{\Delta}\ldots)^{\Delta})^{\Delta}]^{\alpha}\}^{\Delta} + f(t, x(g(t))) = 0,$$

where α is the quotient of odd positive integers, $g:\mathbb{T}\to\mathbb{T}$ with $g(t) > t$ and $\lim_{t\to\infty} g(t) = \infty$ and there there exists a positive rd-continuous function $p(t)$ such that $\frac{f(t,u)}{u^{\alpha}} \geq p(t)$ for $u \neq 0$. Sun et al [5] presented some criteria for oscillation and asymptotic behavior of dynamic equation

$$\{r_{n-1}(t)[(r_{n-2}(t)(\ldots(r_1(t)x^{\Delta}(t))^{\Delta}\ldots)^{\Delta})^{\Delta}]^{\alpha}\}^{\Delta} + f(t, x(g(t))) = 0,$$

where $\alpha \geq 1$ is the quotient of odd positive integers, $g:\mathbb{T}\to\mathbb{T}$ is an increasing differentiable function with $g(t) \leq t$, $g\circ\sigma = \sigma\circ g$ and $\lim_{t\to\infty} g(t) = \infty$ and there there exists a positive rd-continuous function $p(t)$ such that $\frac{f(t,u)}{u^{\beta}} \geq p(t)$ for $u \neq 0$ and $\beta \geq 1$ is the quotient of odd positive integers. Sun et al [6] considered quasilinear dynamic equation of the form

$$\left\{r_{n-1}(t)\left[(r_{n-2}(t)(\ldots(r_1(t)x^{\Delta}(t))^{\Delta}\ldots)^{\Delta})^{\Delta}\right]^{\alpha}\right\}^{\Delta} + p(t)x^{\beta}(t) = 0,$$

where α, β are the quotient of odd positive integers.

Also, The results obtained in [4–6] are given when

$$\int_{t_0}^{\infty}\frac{1}{r_{n-2}(t)}\left\{\int_t^{\infty}\left[\frac{1}{r_{n-1}(s)}\int_s^{\infty}p(u)\Delta u\right]^{1/\alpha}\Delta s\right\}\Delta t = \infty. \tag{1.4}$$

Hassan and Kong [7] obtained asymptotics and oscillation criteria for the nth-order half-linear dynamic equation with deviating argument

$$(x^{[n-1]})^{\Delta}(t) + p(t)\phi_{\alpha[1,n-1]}(x(g(t))) = 0,$$

where $\alpha[1, n-1] := \alpha_1\cdots\alpha_{n-1}$; and Grace and Hassan [8] further studied the asymptotics and oscillation for the higher order nonlinear dynamic equation with Laplacians and deviating argument

$$(x^{[n-1]})^{\Delta}(t) + p(t)\phi_{\gamma}(x^{\sigma}(g(t))) = 0.$$

However, the establishment of the results in [8] requires the restriction on the time scale $\mathbb{T}$ that $g^*\circ\sigma = \sigma\circ g^*$ where $g^*(t) = \min\{t, g(t)\}$ (though it is missed in most places) which is hardly satisfied. For more results on dynamic equations, we refer the reader to the papers [9–14,16,15,17–26].

In this paper, we will discuss the higher order nonlinear dynamic equation (1.1) with Laplacians and deviating argument on a general time scale without any restrictions on $g(t)$ and $\sigma(t)$ and also without the conditions (1.3) and (1.4). Some asymptotics and oscillation criteria will be derived for the cases when n is even and odd, and when $\alpha \geq \gamma$ and $\alpha \leq \gamma$, respectively, with $\alpha = \alpha_1\cdots\alpha_{n-1}$. The results in this paper improve the results in [3–8] on the oscillation of various dynamic equations.

2. Main results

We introduce the following notation:

$$\alpha[h,k] := \begin{cases}\alpha_h\cdots\alpha_k & h \leq k,\\ 1, & h > k,\end{cases} \tag{2.1}$$

with $\alpha = \alpha[1, n-1]$. For any $t, s \in \mathbb{T}$ and for a fixed $m \in \{0, 1, \ldots, n-1\}$, define the functions $R_{m,j}(t, s)$ and $p_j(t)$, $j = 0, 1, \ldots, m$, by the following recurrence formulas:

$$R_{m,j}(t,s) := \begin{cases}1, & j = 0,\\ \int_s^t\left[\frac{R_{m,j-1}(\tau,s)}{r_{m-j+1}(\tau)}\right]^{1/\alpha_{m-j+1}}\Delta\tau, & j = 1,2,\ldots,m,\end{cases} \tag{2.2}$$

and

$$p_j(t) := \begin{cases}p(t), & j = 0,\\ \left[\frac{1}{r_{n-j}(t)}\int_t^{\infty}p_{j-1}(\tau)\Delta\tau\right]^{1/\alpha_{n-j}}, & j = 1,2,\ldots,n-1,\end{cases} \tag{2.3}$$

provided the improper integrals involved are convergent.

In order to prove the main results, we need the following lemmas. The first one is an extension of Lemma 2.1 in [7] to the nonlinear Eq. (1.1) with exactly the same proof.

Lemma 2.1. *Assume Eq. (1.1) has an eventually positive solution $x(t)$. Then there exists an integer $m \in \{0, 1, \ldots, n-1\}$ with $m + n$ odd such that*

$$x^{[k]}(t) > 0 \quad \text{for} \quad k = 0, 1, \ldots, m \tag{2.4}$$

and

$$(-1)^{m+k}x^{[k]}(t) > 0 \quad \text{for} \quad k = m, m+1, \ldots, n-1 \tag{2.5}$$

eventually.

Remark 2.1. If $n = 2$ in Lemma 2.1 then $m = 1$, whereas if $n = 3$ then $m = 2$ or $m = 0$.

Remark 2.2. If $n \geq 4$ in Lemma 2.1 and

$$\int_{t_0}^{\infty} p_2(\tau)\Delta\tau = \infty, \tag{2.6}$$

then

$$m := \begin{cases}n-1, & if \quad n \in 2\mathbb{N},\\ n-1 or 0, & if \quad n \in 2\mathbb{N}-1.\end{cases} \tag{2.7}$$

Proof. From Lemma 2.1 that there exists an integer number $m \in \{0, 1, \ldots, n-1\}$ such that (2.4) and (2.5) hold for $t \geq t_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) $n \in 2\mathbb{N}$. We claim that (2.6) implies that $m = n-1$. In fact, if $1 \leq m \leq n-3$, then for $t \geq t_1$

$$x^{[n]}(t) < 0,\ x^{[n-1]}(t) > 0,\ x^{[n-2]}(t) < 0,\ x^{[n-3]}(t) > 0.$$

Since $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$ then for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x(g(t)) \geq l > 0$ for $t \geq t_2$. It follows that

$$\phi_{\gamma}(x(g(t))) \geq l^{\gamma} \geq L \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}},$$

Eq. (1.1) can be written as

$$-\left(x^{[n-1]}(t)\right)^{\Delta} = p(t)\phi_{\gamma}(x(g(t))) \geq Lp(t) = L\ p_0(t).$$

Integrating the above inequality from t to $v \in [t, \infty)_{\mathbb{T}}$ and letting $v \to \infty$ and using (2.5), we get

$$x^{[n-1]}(t) \geq L\int_t^{\infty} p_0(s)\Delta s,$$

which implies

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq L^{1/\alpha_{n-1}}\left[\frac{1}{r_{n-1}(t)}\int_t^{\infty}p_0(s)\Delta s\right]^{1/\alpha_{n-1}} = L^{1/\alpha_{n-1}}\ p_1(t).$$

By integrating the above inequality from t to $v \in [t, \infty)_{\mathbb{T}}$ and then taking limits as $v \to \infty$ and using the fact $x^{[n-2]} < 0$ eventually, we get

$$-x^{[n-2]}(t) > L^{1/\alpha_{n-1}}\int_t^{\infty}p_1(s)\Delta s,$$