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Original article

An exponential Chebyshev second kind approximation for solving high-order ordinary differential equations in unbounded domains, with application to Dawson's integral

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ABSTRACT

A new exponential Chebyshev operational matrix of derivatives based on Chebyshev polynomials of second kind (ESC) is investigated. The new operational matrix of derivatives of the ESC functions is derived and introduced for solving high-order linear ordinary differential equations with variable coefficients in unbounded domain using the collocation method. As an application the introduced method is used to evaluate Dawson's integral by solving its differential equation. The corresponding differential equation to Dawson's integral is a boundary value problem with conditions tends to infinity. The obtained numerical results are compared with the exact solution and showed good accuracy.

1. Introduction

Spectral methods have been developed rapidly through the last years for the numerical solutions of differential equations. Compared to other numerical methods, spectral methods give high accuracy and have wide range of applications in many mathematical problems and physical phenomena. The main idea of spectral methods is to approximate the solutions of differential equations by means of truncated series of some orthogonal polynomials. The most common spectral methods used to solve ordinary differential equations (ODEs) are tau, collocation, and Galerkin methods. Siyyam [1] used Laguerre tau method to solve ODEs while Parand and Razzaghi [2] used the same method with the same equations but with rational Legendre as the basis function. Guo et al. [3] and Wang et al. [4] employed the Legendre collocation method to solve the initial value problems and Awoyemi and Idowu [5] used the hybrid collocation with third order ODEs. Galerkin method is also applied for solving ODEs [6,7]. Doha et al. used the generalized Jacobi polynomials for solving ODEs [8–11].

Chebyshev polynomials are one of the most important orthogonal polynomials, which are widely used with spectral methods [12].

The Chebyshev first kind $T_n(x)$ are orthogonal polynomials on the finite interval $[-1, 1]$, these polynomials have many applications in numerical analysis [12], and numerous studies show the merits of them in various applications in fluid mechanics. One of the applications of Chebyshev polynomials is the solution of ODEs with initial and boundary conditions, with collocation points [13,14]. Many studies are considered on the finite interval [0, 1] with the help of usual transformation maps the Chebyshev to the shifted Chebyshev polynomial. Therefore, under a transformation that maps the interval $[-1, 1]$ into a semi-infinite domain $[0, \infty)$, several research groups successfully applied spectral methods to solve differential equations [15–26], their transformation maps the Chebyshev polynomials to the rational Chebyshev functions (RC) and defined by.

$$R_n(x) = T_n\left(\frac{x-1}{x+1}\right). \quad (1)$$

Furthermore, Koc and Kurnaz [27] have proposed a modified type of Chebyshev polynomials as an alternative to the solutions of the partial differential equations defined in real domain. In their study, the basis functions called exponential Chebyshev (EC) functions $E_n(x)$ which are orthogonal in $(-\infty, \infty)$. This kind of

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extension tackles the problems over the whole real domain. The EC functions are defined as

$$E_n(x) = T_n\left(\frac{e^x-1}{e^x+1}\right). \tag{2}$$

In our previous report [28] we introduced a modified form of the operational matrix of the derivatives by processing the truncation made by Koc and Kurnaz [27] and applied it to ODEs defined in whole rang. Recently, we reported a new operational matrix of derivatives of EC functions for solving ODEs in unbounded domains [29].

In this paper we introduce a new operational matrix of derivatives based on exponential Chebyshev of the second kind (ESC) functions and employ it to solve ODEs with variable coefficients in unbounded domains using the collocation method.

As an application of our method we find approximate solution to Dawson's integral by solving its differential equation with the subjected condition that tends to infinity. The high-order linear nonhomogeneous differential equations that considered here in this paper is

$$\sum_{k=0}^{m} q_k(x)\ \phi^{(k)}(x) = f(x), \quad -\infty < x < \infty, \tag{3}$$

with the mixed conditions

$$\sum_{k=0}^{m-1}\sum_{j=0}^{J} d_{ij}^{k}\,\phi^{(k)}\left(b_j\right) = \alpha_i,$$
$$-\infty < b_j < \infty, \quad i = 0,1,\ldots,m-1, \quad j = 0,1,\ldots,J \tag{4}$$

where, $q_k(x)$ and $f(x)$ are continuous functions on the interval $(-\infty,\infty)$, d_{ij}^k, b_j and α_i are appropriate constants, or b_j may tends to $\pm\infty$ (the boundary condition tends to infinity).

2. The exponential Chebyshev functions of second kind

In this section we list the definition and some properties of the ESC functions.

2.1. Definition of ESC functions

The ESC function of the form

$$E_n^U(x) = U_n\left(\frac{e^x-1}{e^x+1}\right), \tag{5}$$

where $U_n(x)$ is the Chebyshev polynomials of the second kind which are orthogonal polynomials of degree n in x defined on the interval $[-1,\ 1]$ (see Ref. [12] and [30] for more details).

And the corresponding recurrence relation takes the following form

$$E_0^U(x) = 1, \quad E_1^U(x) = 2\left(\frac{e^x-1}{e^x+1}\right),$$
$$E_{n+1}^U(x) = 2\left(\frac{e^x-1}{e^x+1}\right)E_n^U(x) - E_{n-1}^U(x). \quad n \geq 1 \tag{6}$$

2.2. ESC functions are orthogonal

The ESC functions are orthogonal in the interval $(-\infty,\infty)$ with respect to the weight function $w(x)$ which is given by $4e^{3x/2}(e^x+1)^{-3}$, with the orthogonality condition

$$\left\langle E_n^U(x),\ E_m^U(x)\right\rangle = \int_{-\infty}^{\infty} E_n^U(x)E_m^U(x)w(x)dx = \frac{\pi}{2}\delta_{nm}, \tag{7}$$

where, δ_{nm} is the Kronecker delta function and $\langle *, *\rangle$ is the inner product notation.

Also the product relation of ESC functions is given by

$$\left(\frac{e^x-1}{e^x+1}\right)E_n(x) = \frac{1}{2}[E_{n+1}^U(x) + E_{n-1}^U(x)] \tag{8}$$

2.3. Function expansion in terms of ESC functions

A function $h(x)$ is well defined over the interval $(-\infty,\infty)$ and can be expanded in terms of ESC functions as

$$h(x) = \sum_{i=0}^{\infty} a_i E_i^U(x), \tag{9}$$

where

$$a_i = \frac{2}{\pi}\int_{-\infty}^{\infty} E_i^U(x)h(x)w(x)dx.$$

If the summation in expression (9) is truncated to N where $N < \infty$ it takes the following form

$$h(x) \cong \sum_{i=0}^{N} a_i E_i^U(x), \tag{10}$$

also, the (k)th-order derivative of $h(x)$ can be written as

$$h^{(k)}(x) \cong \sum_{i=0}^{N} a_i\left(E_i^U(x)\right)^{(k)} \tag{11}$$

where $(E_n^U(x))^{(0)} = E_n^U(x)$.

2.4. The operational matrix

The new representation of ESC functions is presented as follows.

The Chebyshev polynomials of first kind $T_n(x)$ can be expressed in terms of x^n in different formulas found in Ref. [12], one of them is

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k 2^{n-2k-1}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}, \quad 2k \leq n. \tag{12}$$

Similar relation found in [30,31] for the Chebyshev polynomials of second kind $U_n(x)$ takes the following form

$$U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k\binom{n-k}{k}(2x)^{n-2k}, \tag{13}$$

by the help of properties of gamma function the previous relation takes the form

$$U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k 2^{n-2k}\frac{\Gamma(n-k+1)}{\Gamma(k+1)\Gamma(n-2k+1)}x^{n-2k}, \quad n > 0, \tag{14}$$

where, $[\frac{n}{2}]$ denotes the integer part of the value $\frac{n}{2}$.

If we use the expression $v(x) = \frac{e^x-1}{e^x+1}$ in the ESC functions, we can express it explicitly in terms of powers of $v(x)$ as

$$E_n^U(x) = \sum_{k=0}^{[n/2]} (-1)^k 2^{n-2k}\binom{n-k}{k}(v(x))^{n-2k}, \tag{15}$$

from previous relation with simple modification we can define:

if n is even number

$$E_n^U(x) = E_{2l}^U(x) = \sum_{j=0}^{l} (-1)^{l-j}2^{2j}\binom{l+j}{l-j}(v(x))^{2j}, \tag{16}$$