Journal of the Egyptian Mathematical Society 000 (2016) 1–9

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original Article

On the dynamics of a higher order rational difference equations

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A R T I C L E I N F O

Article history:
Received 24 November 2015
Revised 19 January 2016
Accepted 29 June 2016
Available online xxx

MSC:
39A10
39A11
39A99
34C99

Keywords:
Difference equations
Stability
Global stability
Boundedness
Periodic solutions

A B S T R A C T

The main objective of this paper is to study the global stability of the positive solutions and the periodic character of the difference equation

$$y_{n+1} = ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k} + ey_{n-s}}{\alpha y_{n-k} + \beta y_{n-s}}, \quad n = 0, 1, \ldots,$$

with positive parameters and non-negative initial conditions. Numerical examples to the difference equation are given to explain our results.

1. Introduction

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the periodic character, the boundedness character and the global behaviors of their solutions. The study of non-linear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. It is worthwhile to point out that although several approaches have been developed for finding the global character of difference equations, relatively a large number of difference equations has not been thoroughly understood yet [1–21].

In recent years non-linear difference equations have attracted the interest of many researchers, for example:

Kalabušić et al. [12] investigated the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the sequence p_n is periodic with period $k_2 = \{2, 3\}$ with positive terms and the initial conditions are positive.

Raafat [15] studied the global attractivity, periodic nature, oscillation and the boundedness of all admissible solutions of the difference equations

$$x_{n+1} = \frac{A - Bx_{n-1}}{\pm C + Dx_{n-2}}, \quad n = 0, 1, \ldots,$$

where A, B are non-negative real numbers, C, D are positive real numbers $\pm C + Dx_{n-2} \neq 0$ and for all $n \geq 0$.

Alaa [16] investigated the global stability, the permanence, and the oscillation character of the recursive sequence

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots,$$

where α is a negative number and the initial conditions x_{-1} and x_0 are negative numbers.

Obaid et al. [17] investigated the global stability character, boundedness and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2} + dx_{n-3}}{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}},$$

where the parameters a, b, c, d, α, β and γ are positive real numbers and the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are positive real numbers.

In [18] Zayed studied the global stability and the asymptotic properties of the non-negative solutions of the non-linear

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http://dx.doi.org/10.1016/j.joems.2016.06.010
1110-256X/

difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}, \qquad n = 0, 1, \ldots,$$

where the parameters A, B, p, q and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers, while k is a positive integer number.

El-Moneam [19] got the periodicity, the boundedness and the global stability of the positive solutions of the non-linear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}},$$
$$n = 0, 1, \ldots,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma}, \ldots, x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$.

Our aim in this paper is to study some qualitative behavior of the positive solutions of the difference equation

$$y_{n+1} = ay_n + by_{n-t} + cy_{n-l} + \frac{dy_{n-k} + ey_{n-s}}{\alpha y_{n-k} + \beta y_{n-s}}, \ n = 0, 1, \ldots, \qquad (1)$$

where the initial conditions $x_{-\delta}, x_{-\delta+1} \ldots, x_{-1}$ and x_0 are positive real numbers where $\delta = max\{t, l, k, s\}$ and the coefficients a, b, c, d, e, α and β are positive real numbers.

2. Some basic definition

Let I be some interval of real numbers and let

$$F : I^{\delta+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-\delta}, x_{-\delta+1}, \ldots, x_0 \in I$, the difference equation

$$y_{n+1} = F(y_n, y_{n-1}, \ldots, y_{n-\delta}), \quad n = 0, 1, \ldots, \qquad (2)$$

has a unique solution $\{y_n\}_{n=-\delta}^{\infty}$.

Definition 1 (Equilibrium Point)**.** A point $\bar{y} \in I$ is called an equilibrium point of the difference Eq. (2) if

$$\bar{y} = F(\bar{y}, \bar{y}, \ldots, \bar{y}).$$

That is, $y_n = \bar{y}$ for $n \geq 0$, is a solution of the difference Eq. (2), or equivalently, $\bar{y}$ is a fixed point of F.

Definition 2 (Stability)**.** Let $\bar{y} \in (0, \infty)$ be an equilibrium point of the difference Eq. (2). Then, we have

(i) The equilibrium point $\bar{y}$ of the difference Eq. (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $y_{-\delta}, \ldots, y_{-1}, y_0 \in I$ with

$$|y_{-\delta} - \bar{y}| + \ldots + |y_{-1} - \bar{y}| + |y_0 - \bar{y}| < \delta,$$

we have

$$|y_n - \bar{y}| < \epsilon \quad \text{for all} \quad n \geq -\delta.$$

(ii) The equilibrium point $\bar{y}$ of the difference Eq. (2) is called locally asymptotically stable if $\bar{y}$ is locally stable solution of Eq. (2) and there exists $\gamma > 0$, such that for all $y_{-\delta}, \ldots, y_{-1}, y_0 \in I$ with

$$|y_{-\delta} - \bar{y}| + \ldots + |y_{-1} - \bar{y}| + |y_0 - \bar{y}| < \gamma,$$

we have

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(iii) The equilibrium point $\bar{y}$ of the difference Eq. (2) is called global attractor if for all $y_{-\delta}, \ldots, y_{-1}, y_0 \in I$, we have

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(iv) The equilibrium point $\bar{y}$ of the difference Eq. (2) is called globally asymptotically stable if $\bar{y}$ is locally stable, and $\bar{y}$ is also a global attractor of the difference Eq. (2).

(v) The equilibrium point $\bar{y}$ of the difference Eq. (2) is called unstable if $\bar{y}$ is not locally stable.

Definition 3 (Periodicity)**.** A sequence $\{y_n\}_{n=-\delta}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -\delta$. A sequence $\{y_n\}_{n=-\delta}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 4. Eq. (2) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $y_{-\delta}, \ldots, y_{-1}, y_0 \in (0, \infty)$ there exists a positive integer N which depends on these initial conditions such that

$$m \leq y_n \leq M \quad \text{for all } n > N.$$

Definition 5. The linearized equation of the difference Eq. (2) about the equilibrium $\bar{y}$ is the linear difference equation

$$x_{n+1} = \sum_{i=0}^{\delta} \frac{\partial F(\bar{y}, \bar{y}, \ldots, \bar{y})}{\partial y_{n-i}} x_{n-i}. \qquad (3)$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0\lambda^{\delta} + p_1\lambda^{\delta-1} + \ldots + p_{\delta-1}\lambda + p_{\delta} = 0, \qquad (4)$$

where

$$p_i = \frac{\partial F(\bar{y}, \bar{y}, \ldots, \bar{y})}{\partial y_{n-i}}.$$

Theorem 1 [3]**.** *Assume that $p_i \in R$, $i = 1, 2, \ldots, \delta$ and δ is non-negative integer. Then*

$$\sum_{i=1}^{\delta} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+\delta} + p_1 x_{n+\delta-1} + \ldots + p_{\delta} x_n = 0, \quad n = 0, 1, \ldots .$$

Theorem 2 [4]**.** *Let $g : [\eta, \xi]^{\delta+1} \to [\eta, \xi]$, be a continuous function, where δ is a positive integer, and where $[\eta, \xi]$ is an interval of real numbers. Consider the difference equation*

$$y_{n+1} = g(y_n, y_{n-1}, \ldots, y_{n-\delta}), \qquad n = 0, 1, \ldots . \qquad (5)$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq \delta + 1$; the function $g(z_1, z_2, \ldots, z_{\delta+1})$ is weakly monotonic in z_i that is if $z_i \geq \acute{z}_i$ then $g(z_1, z_2, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_{\delta+1}) > g(z_1, z_2, \ldots, z_{i-1}, \acute{z}_i, z_{i+1}, \ldots, z_{\delta+1})$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, \ldots, m_{\delta+1}), \qquad M = g(M_1, M_2, \ldots, M_{\delta+1}),$$

then $m = M$, where for each $i = 1, 2, \ldots, \delta + 1$, we set

$$m_i = \begin{cases} m, & \text{if } g \text{ is non-decreasing in } z_i \\ M, & \text{if } g \text{ is non-increasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M, & \text{if } g \text{ is non-decreasing in } z_i \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium point $\bar{y}$ of Eq. (5), and every solution of Eq. (5) converges to $\bar{y}$.