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Solutions of fractional order electrical circuits via Laplace transform and nonstandard finite difference method

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In this article, fractional linear electrical systems are investigated. Analytical solutions of the fractional models are derived using Laplace transform method. Also, numerical simulations using Grünwald–Letnikov definition are proposed. Comparisons between fractional and classical electrical systems are illustrated using Laplace transform and nonstandard finite difference method.

1. Introduction

In recent years, fractional calculus has an interest to mathematicians as it has many engineering applications [1–9,28–31]. It provides excellent instruments for the description of memory and properties of various materials and processes. Non-integer derivatives play an important role in modeling of electrical circuits that contains super capacitors and super inductors. Moreover, in such electrical circuits, singular linear systems were addressed in many papers and books [10–13]. The charging and discharging processes of different capacitors in R-C electrical circuits theoretically and experimentally are considered in [14]. Also the authors investigated the nonlocal behavior in these processes that arising from the time fractality via fractional calculus. The existence and uniqueness of the solution of an RLC circuit model were discussed and the solution of that model was obtained by Adomian Decomposition Method (ADM) and Laplace Transform method in [15]. The solution of a new class of singular fractional electrical circuits using Weierstrass regular pencil decomposition and Laplace transform are proposed in [16]. In this article, we will investigate analytical and numerical solutions for both R-L and R-C electrical circuit models using Laplace transform method and nonstandard finite difference methods (NSFDM).

This article is organized as: basic definitions and some properties of fractional calculus are given in Section 2. In Section 3 analytical solutions for different electrical circuits with fractional order derivatives and numerical solution using NSFDMs are derived, while some illustrative examples with their solutions are given in Section 4. Finally a brief conclusion is given in Section 5.

2. Preliminaries

In this section, we introduce some basic definitions and functions that have important rules in fractional calculus which are further used in this article, [17–25].

Definition 2.1. Caputo fractional derivative

The fractional derivative of a function $f(x)$ of non-integer order α is given as,

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$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-1)^{n-\alpha-1} \frac{d^n}{dt^n} f(t)dt, n-1<\alpha<n. \tag{2.1}$$

Definition 2.2. Riemann–Liouville fractional derivative

The fractional derivative of the function $f(x)$ is given as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt, \ \alpha>0. \tag{2.2}$$

Definition 2.3. Grünwald–Letnikov fractional derivative

In 1867, Grünwald–Letnikov defined the fractional derivative of a function $f(x)$ as:

$$D^{\alpha} f(x) = \lim_{N\to\infty} \frac{1}{h^{\alpha}} \sum_{j=0}^{N} C_j^{\alpha} f(x-jh), n-1<\alpha<n, \ h=\frac{1}{N}, \tag{2.3}$$

where $C_0^{\alpha}=1$ and $C_j^{\alpha} = (1-\frac{1+\alpha}{j})C_{j-1}^{\alpha}$.

The Caputo fractional derivative is not equivalent to the Riemann– Liouville fractional derivative and they are related by $D^{\alpha} f(t) = {}^{R}D^{\alpha}(f(t)-f(0))$ for $0<\alpha<1$. If the initial condition $f(0)=0$, then we have that $D^{\alpha} f(t) = {}^{R}D^{\alpha} f(t)$ and the Grünwald–Letnikov fractional derivatives is equivalent to the Caputo fractional derivative, see [19,30].

Definition 2.4. Laplace Transform

If a function $f(t)$ is of exponential order α and is a piece-wise continuous on real line, then Laplace transform of $f(t)$ for $s>\alpha$ is defined by:

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt. \tag{2.4}$$

Laplace transform of Caputo derivative is:

$$L[D^{\alpha} f(t)] = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \ f^{(k)}(0), \quad n=\lceil \alpha \rceil. \tag{2.5}$$

Laplace transform of the convolution of two functions:

The convolution of two functions $f(t)$ and $g(t)$ is defined by:

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)\ d\tau, \tag{2.6}$$

and the Laplace transform of the convolution of two functions $f(t), g(t)$ is defined by:

$$L\{f(t)*g(t)\} = L\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} = F(s)G(s), \tag{2.7}$$

where $F(s)$ and $G(s)$ are the Laplace transform of $f(t)$ and $g(t)$ respectively.

In this article, we will generalize some electrical circuits models to a fractional order system of order α in sense of caputo definition. We assume that the voltage across the inductor v_l and the capacitor current i_c are:

$$v_l = l\ \frac{d^{\alpha} i_l}{dt^{\alpha}}, \tag{2.8}$$

$$i_c = c\frac{d^{\alpha} v_c}{dt^{\alpha}}, \tag{2.9}$$

where $\frac{d^{\alpha}}{dt^{\alpha}} = D^{\alpha}$ is the fractional derivative operator in the sense of Caputo derivative. Also l is the inductance, c is the capacitance and i_l *and* v_c are the inductor current and the capacitor voltage respectively.

Definition 2.5. The function $E_t(\alpha, a)$ [18,19]

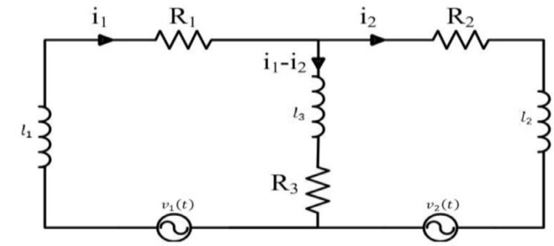

Fig. 1. Electrical circuit of application 1.

The function $E_t(\alpha, a)$ is a solution of the ordinary differential equation

$$(D-a)\ y = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ Re(\alpha)>0 \tag{2.10}$$

and it is defined by:

$$E_t(\alpha, a) = t^{\alpha}\ e^{a\ t}\ \gamma^*(\alpha, a\ t), \tag{2.11}$$

where $\gamma^*(\alpha, a\ t)$ is the incomplete gamma function defined in [19] as

$$\gamma^*(\alpha, z) = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha+k+1)}. \tag{2.12}$$

If we replace a by ia in (2.11), then

$$E_t(\alpha, ia) = C_t(\alpha, a) + i\ S_t(\alpha, a), \tag{2.13}$$

where $C_t(\alpha, a) = t^{\upsilon}\ \sum_{k\ even}^{\infty} \frac{(-1)^{k/2}(a\ t)^k}{\Gamma(\alpha+k+1)}$, and $S_t(\alpha, a) = t^{\alpha}\ \sum_{k\ odd}^{\infty} \frac{(-1)^{(k-1)/2}\ (a\ t)^k}{\Gamma(\alpha+k+1)}$.

The function $S_t(\alpha, a)$ or $C_t(\alpha+1, a)$ is a solution of the following ordinary differential equation

$$\left(D^2+a^2\right)\ y = \frac{a\ t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha>0.$$

3. Fractional linear electrical systems

In this section, we will introduce some applications for some electrical circuits with fractional order $\alpha, 0<\alpha\le 1$. Analytical solutions are briefly obtained for each application.

3.1. Analytical solutions of fractional systems

Firstly, we consider the following applications:

Application 1. Consider the electrical circuit shown in Fig. 1 with given resistances R_1, R_2, R_3, inductances l_1, l_2, l_3 and voltage sources $v_i(t),\ i=1,2$.

Using Kirchhoff's voltages and currents laws and considering (2.8) and (2.9) we get:

$$v_1(t) = i_1R_1 + l_1\frac{d^{\alpha}i_1}{dt^{\alpha}} + l_3\frac{d^{\alpha}(i_1-i_2)}{dt^{\alpha}} + \ R_3(i_1-i_2), \tag{3.1}$$

$$v_2(t) + R_3(i_1-i_2) + l_3\frac{d^{\alpha}(i_1-i_2)}{dt^{\alpha}} = i_2R_2 + l_2\frac{d^{\alpha}i_2}{dt^{\alpha}}. \tag{3.2}$$

We can write Eqs. (3.1) and (3.2) in the following form

$$\begin{bmatrix} l_1+l_3 & -l_3 \\ l_3 & -(l_2+l_3) \end{bmatrix} \frac{d^{\alpha}}{dt^{\alpha}} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} -(R_1+R_3) & R_3 \\ -R_3 & (R_2+R_3) \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ -v_2(t) \end{bmatrix}. \tag{3.3}$$

Let $q = \begin{bmatrix} l_1+l_3 & -l_3 \\ l_3 & -(l_2+l_3) \end{bmatrix}, I(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}, R = \begin{bmatrix} -(R_1+R_3) & R_3 \\ -R_3 & (R_2+R_3) \end{bmatrix}, V(t) = \begin{bmatrix} v_1(t) \\ -v_2(t) \end{bmatrix}$, and $q^{-1} = \frac{-1}{l_1l_2+l_1l_3+l_2l_3}\begin{bmatrix} -(l_2+l_3) & l_3 \\ -l_3 & l_1+l_3 \end{bmatrix}$ is the inverse of q.