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Review Paper

Smarandache curves in the Galilean 4-space G_4

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ABSTRACT

In this paper, we study Smarandache curves in the 4-dimensional Galilean space G_4. We obtain Frenet-Serret invariants for the Smarandache curve in G_4. The first, second and third curvature of Smarandache curve are calculated. These values depending upon the first, second and third curvature of the given curve. Examples will be illustrated.

1. Introduction

Galilean space is the space of the Galilean Relativity. For more about Galilean space and pseudo Galilean space may be found in [1–3]

The geometry of the Galilean Relativity acts like a bridge from Euclidean geometry to special Relativity. The geometry of curves in Euclidean space have been developed a long time ago [4]. In recent years, mathematicians have begun to investigate curves and surfaces in Galilean space [5].

Galilean space is one of the Cayley-Klein spaces. Smarandache curves have been investigated by some differential geometers such as H.S. Abdelaziz, M. Khalifa and Ahmad T. Ali [6]. In this paper, we study Smarandache curve in 4-dimensional Galilean space G_4 and characterize such curves in terms of their curvature functions.

2. Preliminaries

The three-dimensional Galilean space G_3, is the Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$. The absolute of the Galilean geometry is an ordered triple (ω, f, I) where ω is the ideal (absolute) plane, f is a line in ω (absolute line) and I is elliptic involution point $(0, 0, x_2, x_3) \to (0, 0, x_3, -x_2)$.

A plane is called Euclidean if it contains f, otherwise it is called isotropic or, i.e. planes $x = const.$ are Euclidean, and so is the plane ω . A vector $u = (u_1, u_2, u_3)$ is said to be non-isotropic vector if $u_1 \neq 0$. all unit non-isotropic vectors are of the form $u = (1, u_2, u_3)$. For isotropic vectors $u_1 = 0$ holds.

In the Galilean space G_3 there are four classes of lines [7]:

1. The (proper) isotropic lines that don't belong to the plane ω but meet the absolute line *f*.
2. The (proper) non-isotropic lines they don't meet the absolute line *f*.
3. a proper non-isotropic lines all lines of ω but *f*.
4. The absolute line *f*.

Let $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$ and $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ be two vectors in G_4. The Galilean scalar product in G_4 can be written as

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle_{G_4} = \begin{cases} x_1 y_1 & \text{if } x_1 \neq 0 \text{ and } y_1 \neq 0 \\ x_2 y_2 + x_3 y_3 + x_4 y_4 & \text{if } x_1 = 0 \text{ or } y_1 = 0 \end{cases}$$

The norm of the vector $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$ is defined by $|\overrightarrow{x}|_{G_4} = \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle_{G_4}}$.

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The Galilean cross product of the vectors x, y, z on G_4 is defined by

$$\vec{x}\times\vec{y}\times\vec{z} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0 \text{ or } z_1 \neq 0 \\ \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} & \text{if } x_1 = y_1 = z_1 = 0 \end{cases}$$

where $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, and $e_4 = (0,0,0,1)$

The Galilean G_4 studies all properties invariant under motions of objects in space is even more complex. In addition, it stated this geometry can described more precisely as the study of those properties of 4D space with coordinate which are invariant under general Galilean transformation as follows [8].

$$\begin{aligned} x' &= (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)x \\ &\quad + (\sin\beta\cos\alpha - \cos\gamma\cos\beta\sin\alpha)y \\ &\quad + (\sin\gamma\sin\alpha)z + (v\cos\delta_1)t + a \\ y' &= -(\cos\beta\sin\alpha + \cos\gamma\sin\beta\cos\alpha)x \\ &\quad + (-\sin\beta\sin\alpha + \cos\gamma\cos\beta\cos\alpha)y \\ &\quad + (\sin\gamma\cos\alpha)z + (v\cos\delta_2)t + b \\ z' &= (\sin\gamma\sin\beta)x - (\sin\gamma\cos\beta)y \\ &\quad + (\cos\gamma)z + (v\cos\delta_3)t + c \\ t' &= t + d \end{aligned}$$

with $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1$.

A curve $\alpha\colon I \to G_4$ of C^∞, $I \subset R$ in the Galilean G_4 is defined by $\alpha(s) = (s, y(s), z(s), w(s))$ where the curve α is parameterized by the Galilean invariant arc-length. The first Frenet-Serret frame , that is, the tangent vector of $\alpha(s)$ in G_4, is defined by

$$t(s) = \alpha'(s) = (1, y'(s), z'(s), w'(s)) \tag{2.1}$$

The second vector of the Frenet-Serret frame , that is called, the principle normal of $\alpha(s)$ is defined by $n(s)$.

$$n(s) = \frac{1}{k_1(s)}\alpha''(s) = \frac{1}{k_1(s)}(0, y''(s), z''(s), w''(s)) \tag{2.2}$$

The third vector of the Frenet-Serret frame , that is called, the first binormal vector of is defined by

$$b_1(s) = \frac{1}{k_2(s)}\left(0, \left(\frac{y''(s)}{k_1(s)}\right)', \left(\frac{z''(s)}{k_1(s)}\right)', \left(\frac{w''(s)}{k_1(s)}\right)'\right) \tag{2.3}$$

Thus the vector $b_1(s)$ is perpendicular to both $t(s)$ and $n(s)$.

The second binormal vector of $\alpha(s)$ which is the fourth vector of the Frenet-Serret frame is defined by $b_2(s)$.

$$b_2(s) = t(s) \times n(s) \times b_1(s) \tag{2.4}$$

where $k_1(s)$, $k_2(s)$ and $k_3(s)$ are the first, second and third curvature functions of the curve $\alpha(s)$ which are defined by

$$\begin{aligned} k_1(s) &= \left|t'(s)\right|_{G_4} = \sqrt{(y''(s))^2 + (z''(s))^2 + (w''(s))^2} \\ k_2(s) &= \left|n'(s)\right|_{G_4} = \sqrt{\langle n', n'\rangle_{G_4}} \\ k_3(s) &= \langle b_1'(s), b_2(s)\rangle_{G_4} \end{aligned}$$

If the curvature $k_1(s)$, $k_2(s)$ and $k_3(s)$ are constants, then the curve $\alpha(s)$ is called W-curve. The set $\{t(s), n(s), b_1(s), b_2(s), k_1(s), k_2(s), k_3(s)\}$ is called the Frenet-Serret pparatus of the curve α.

The vectors $\{t(s), n(s), b_1(s), b_2(s)\}$ are mutually orthogonal vectors

$$\begin{aligned} \langle t(s), t(s)\rangle_{G_4} &= \langle n(s), n(s)\rangle_{G_4} = \langle b_1(s), b_1(s)\rangle_{G_4} \\ &= \langle b_2(s), b_2(s)\rangle_{G_4} = 1 \end{aligned}$$

and

$$\begin{aligned} \langle t(s), n(s)\rangle_{G_4} &= \langle t(s), b_1(s)\rangle_{G_4} = \langle t(s), b_2(s)\rangle_{G_4} \\ &= \langle n(s), b_1(s)\rangle_{G_4} = \langle n(s), b_2(s)\rangle_{G_4} \\ &= \langle b_1(s), b_2(s)\rangle_{G_4} = 0 \end{aligned}$$

The derivatives of the Frenet-Serret equations are defined as in [9].

$$\begin{aligned} t'(s) &= k_1(s)n(s) \\ n'(s) &= k_2(s)b_1(s) \\ b_1'(s) &= -k_2(s)n(s) + k_3(s)b_1(s) \\ b_2'(s) &= -k_3(s)b_1(s) \end{aligned}$$

3. tb_2 Smarandache curves in G_4

Definition 1. A curve in G_4, whose position vector is obtained by Frenet frame vectors on another curve, is called Smarandache curve.

Let us define special forms of Smarandache curves.

Definition 2. Let $\alpha(s)$ be a unit speed curve in G_4 with constant curvatures k_1, k_2 and k_3 and $\{t(s), n(s), b_1(s), b_2(s)\}$ be Frenet frame on it. The tb_2 Smarandache curves are defined by

$$\beta(s_\beta(s)) = \frac{1}{\sqrt{2}}(t(s) + b_2(s))$$

Theorem 1. *Let $\alpha = \alpha(s)$ be a unit speed curve with constant curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ and $\beta(s_\beta(s))$ be tb_2 Smarandache curve defined by frame vectors of $\alpha(s)$, then*

$$\begin{aligned} t_\beta(s_\beta(s)) &= \frac{k_1 n - k_3 b_1}{\sqrt{k_1^2 + k_3^2}} \\ n_\beta(s_\beta(s)) &= \frac{k_2 k_3 n + k_1 k_2 b_1 - k_3^2 b_2}{\sqrt{k_2^2 k_3^2 + k_1^2 k_2^2 + k_3^4}} \\ b_{1\beta}(s_\beta(s)) &= \frac{(-k_1 k_2^2 n + k_3(k_2^2 + k_3^2)b_1 + k_1 k_2 k_3 b_2)}{\sqrt{k_2^2 + k_3^2}\sqrt{k_2^2 k_3^2 + k_1^2 k_2^2 + k_3^4}} \\ b_{2\beta}(s_\beta(s)) &= \frac{k_1 k_3(k_1^2 k_2^2 + k_3^4 - k_2^2 k_3^2)t}{\sqrt{k_2^2 + k_3^2}\sqrt{k_1^2 + k_3^2}(k_2^2 k_3^2 + k_1^2 k_2^2 + k_3^4)} \\ k_{1\beta}(s_\beta(s)) &= \frac{\sqrt{2}\sqrt{k_2^2 k_3^2 + k_1^2 k_2^2 + k_3^4}}{k_1^2 + k_3^2} \\ k_{2\beta}(s_\beta(s)) &= \frac{\sqrt{2}\sqrt{k_2^2 + k_3^2}}{\sqrt{k_1^2 + k_3^2}} \\ k_{3\beta}(s_\beta(s)) &= 0 \end{aligned}$$

Proof. Let $\beta = \beta(s_\beta(s))$ be a tb_2 Smarandache curve of the curve $\alpha(s)$. Then

$$\beta = \beta(s_\beta(s)) = \frac{1}{\sqrt{2}}(t(s) + b_2(s))$$

$$\begin{aligned} \beta'(s_\beta) &= \frac{d\beta(s_\beta(s))}{ds} = \frac{d\beta(s_\beta)}{ds_\beta}\cdot\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}(t'(s) + b_2'(s)) \\ &= \frac{1}{\sqrt{2}}(k_1 n - k_3 b_1) \end{aligned}$$