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ORIGINAL ARTICLE

On classifications of rational sextic curves

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Abstract In this paper, we extend the Yang's list of reduced sextic plane curves to rational irreducible projective plane curves of type $(6, 3, 1)$.

MATHEMATICS SUBJECT CLASSIFICATION: 14H45; 14R20; 14H30; 14H50

1. Introduction

The determination of all possible configurations of plane algebraic curves for a given degree d is one of the classical and interesting problems in algebraic geometry. Throughout this paper, we work over the field of the complex numbers $\mathbb{C}$. We denote by $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ the projective plane over the field of the complex numbers. The genus is a geometric invariant associated with the curve C, and in the case of $C \subset \mathbb{P}^2$ by a Theorem of Noether (see for instant [1] page 614 or [2] page 222) can be computed as

$$g = \frac{(d-1)(d-2)}{2} - \sum_{P \in \mathrm{Sing}(C)} \frac{m_P(m_P - 1)}{2},$$

where $\mathrm{Sing}(C)$ is the singular points of the curve C and m_P denotes the multiplicity of the singularities of $P \in C$ (including the infinitely near points of P). This invariant plays a very important role in algebraic geometry. For instance, plane curves C with $g = 0$ are called rational curves. In case $g = 1$, 2, C are called elliptic and hyperelliptic curves, respectively. Also, by the genus formula, we easily see that, the lines and the conics have no singular points and an irreducible cubic has at most one double point.

Yoshihara in [3–5], classified plane curves with small degrees whose singular points are only cusps. Curves of degrees $d = 4$, 5 and 6 are called quartic, quintic and sextic curves, respectively. We focus in this paper on a very important type of curves that is irreducible rational projective plane sextic curves.

Let $P \in C$ be a singular point, and let r_P be the number of the branches of C at P. Put $\imath(C) = \sum_{P \in \mathrm{Sing}(C)} (r_P - 1)$. The notation $(d, \nu, \imath)$ is used for curves of degree d, maximal multiplicity of the singularities ν and $\imath = \imath(C)$. For $r_P = 1$, P is called a cusp. In case $r_P \geqslant 2$, Saleem in [7], introduced the notion of the system of the multiplicity sequences of the branches of the curve C at P which explains after how many times of blowing ups of C at P the branches separate from each other.

Yang in [6], gave a list of reduced sextic curves. In his list, he showed the existence of the configurations of these curves. Here, we extend to give a list of irreducible rational projective sextic curves of type $(6, 3, 1)$.

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In [8], Sakai and Saleem classified all rational plane curves of type $(d, d-2)$, possibly with multibranched singularities. They generalized the results with Tono in [9] to plane curves of type $(d, d-2)$ with any genus. It turns out that still the answer of Matsuka and Sakai's conjectured in [10], is affirmative. As a generalization of these results, we have the following question: Is any rational plane curve of type $(d, d-3)$ is transformable into a line by a Cremona transformation? The cuspidal case has been already discussed and answered affirmatively by Flenner-Zaidenberg in [11,12] and Fenske in [13]. In this paper, we answer the question for some classes of rational plane curves of types $(6, 3, 1)$.

For constructing curve germs, Sakai and Tono in [14] used the quadratic Cremona transformations $\varphi_c : (x, y, z) \longrightarrow (xy, y^2, x(z-cx))$ for $c \in \mathbb{C}$, where (x, y, z) are homogeneous coordinates on $\mathbb{P}^2$.

2. Preliminaries

In this section, we investigate a tool for constructing plane curves which is the Cremona transformations $\varphi_c : (x, y, z) \longrightarrow (xy, y^2, x(z-cx))$ for $c \in \mathbb{C}$.

2.1. Singularities on plane curves

Let $(C, P) \subset (\mathbb{C}^2, P)$ be a plane curve germ, where $P \in C$ is a singular point. We obtain the minimal embedded resolution of the singularity (C, P), by means of a sequence of blowing-ups $X_i \xrightarrow{\pi_i} X_{i-1}, i = 1, 2, \ldots, k$, over P. Let $C^{(i)} \subset X_i$ be the strict (also called proper) transform of C in X_i and E is the exceptional divisor of the whole resolution. Hence, the total transform of C in X_k is a simple normal crossing (SNC) divisor $D = E + C^{(k)}$ as in the following diagram:

$$C^{(k)} \xrightarrow{\pi_k} C^{(k-1)} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} C^{(1)} \xrightarrow{\pi_1} C = C^{(0)},$$

where k is a finite positive integer. We recall the properties of the multiplicity sequence $\underline{m}_P(C) = (m_0, m_1, \ldots, m_k)$ of (C, P). Let m_i be the multiplicity of $C^{(i)}$ at P_i, where P_i is the infinitely near point of P on $C^{(i)}$. We define the *multiplicity sequence* of (C, P) to be $\underline{m}_P(C) = (m_0, m_1, \ldots, m_k)$, where $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_k = 1$. For the sequence $\left(\overbrace{m, \ldots, m}^{a\text{-times}}, 1, 1\right)$, we write (m_a).

Here, we recall the definition of *the system of the multiplicity sequences of* $P \in C$ in case the number of the branches of C at P equals 2, (see [7,8]for more details).

Definition 1. The systems of the multiplicity sequences of two branches are defined as follows:

$$\underline{m}_P(\zeta_1, \zeta_2) = \left\{ \begin{pmatrix} m_{1,0} \\ m_{2,0} \end{pmatrix} \cdots \begin{pmatrix} m_{1,\rho} \\ m_{2,\rho} \end{pmatrix} \begin{matrix} m_{1,\rho+1}, m_{1,\rho+2}, \ldots, m_{1,s_1} \\ m_{2,\rho+1}, m_{2,\rho+2}, \ldots, m_{2,s_2} \end{matrix} \right\},$$

where the brackets mean that the germs go through the same infinitely near points of P and $\underline{m}_P(\zeta_i) = (m_{i,0}, m_{i,1}, \ldots, m_{i,s_i})$ are the multiplicity sequences of the branches $(\zeta_i, P), i = 1, 2$, of the germ (C, P).

For a classification of a bibranched singular point Q with multiplicity $d-3$, we give the following proposition.

Proposition 1 [7]. *Let C be a rational plane curve of type $(d, d-3)$. Let $Q \in C$be a bibranched singular point with multiplicity $d-3$. Then, the system of the multiplicity sequences of Q are divided into the following two types $(r, s, v, k > 0$, $i, j \geqslant 0)$:*

(1) *Branches with the same tangent line*	(2) *Branches with different tangent lines*	
$\left\{\begin{pmatrix} k \\ k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{k+j}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 2r \end{pmatrix}\begin{matrix} 2_{k-1} \\ 2_{r+i} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 2k \\ 3r+2 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r, 2 \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ k \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2_i}_{k+j}\right\}$	$\left\{\begin{pmatrix} 2k \\ 2r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 2_{r+i} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r \end{pmatrix}\begin{matrix} 3_{k+v} \\ 3_{r+s} \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ k \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{k+j}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r \end{pmatrix}\begin{matrix} 2_{k-1} \\ 3_{r+s} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_{r+s} \end{matrix}\right\}$
$\left\{\begin{pmatrix} k \\ k+r \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{k}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_{r+s}, 2 \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_{r+s}, 2 \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ k+r \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2j}_{k}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r+1 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+1 \end{pmatrix}\begin{matrix} 3_{k+v} \\ 3_r \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k \\ r \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2_{k+j-r}}_{r}\right\}$	$\left\{\begin{pmatrix} 2k-1 \\ 3r+2 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r, 2 \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+1 \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_r \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k-1 \\ r \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}^{2_{k-r-1}}_{r}\right\}$	$\left\{\begin{pmatrix} 2k \\ 3r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_{r+s} \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+2 \end{pmatrix}\begin{matrix} 3_{k+v} \\ 3_{r+s}, 2 \end{matrix}\right\}$
$\left\{\begin{pmatrix} 2k-1 \\ k+j \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{k-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} 2k \\ 3r \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_{r+s}, 2 \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k \\ 3r+2 \end{pmatrix}\begin{matrix} 3_{k+v}, 2 \\ 3_{r+s}, 2 \end{matrix}\right\}$
	$\left\{\begin{pmatrix} 2k \\ 3r+1 \end{pmatrix}\begin{matrix} 2_{k+j} \\ 3_r \end{matrix}\right\}$	$\left\{\begin{pmatrix} 3k+1 \\ 3r+2 \end{pmatrix}\begin{matrix} 3_k \\ 3_r, 2 \end{matrix}\right\}$

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