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Some results associated with the max–min and min–max compositions of bifuzzy matrices

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Abstract In this paper, we define some kinds of bifuzzy matrices, the max–min (∘) and the min–max (∗) compositions of bifuzzy matrices are defined. Also, we get several important results by these compositions. However, we construct an idempotent bifuzzy matrix from any given one through the min–max composition.

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1. Introduction

The concept of bifuzzy sets (or intuitionistic fuzzy sets) was introduced by Atanassov [1] as a generalization of fuzzy subsets. Later on, much fundamental works have done with this concept by Atanassov [2,3] and others [4–7]. A bifuzzy relation is a pair of fuzzy relations, namely, a membership and a non-membership function, which represent positive and negative aspects of the given information. This is why the concept of bifuzzy relations is a generalization of the idea of fuzzy relations. The name "bifuzzy relations" is used for objects introduced by Atanassov and originally called intuitionistic fuzzy relations (see [1,2]). Bifuzzy relations are also called by some authors "bipolar fuzzy relations" (see [6]). Since the concept of bifuzzy relations is an extension for the concept of ordinary fuzzy relations, the concept of bifuzzy matrices (which represent finite bifuzzy relations) is also an extension for the concept of ordinary fuzzy matrices.

In this paper, we study and prove some properties of bifuzzy matrices throughout some compositions of these matrices. However, we concentrate our attention for the two compositions ∘ (*max–min*) and its dual composition ∗ (*min–max*). We use the definitions of some kinds of bifuzzy matrices such as nearly constant, symmetric, nearly irreflexive and others to prove some results. One of these results enables us to construct an idempotent bifuzzy matrix from any bifuzzy matrix and this is the main result in our work. We also state the relationship between the two compositions ∘ and ∗ of bifuzzy matrices.The motivation for this paper is to study some kinds of finite bifuzzy relations

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throughout bifuzzy matrices by using the two compositions $\circ$ and $*$.

2. Preliminaries and definitions

In system models which based on fuzzy sets, one often uses fuzzy matrices (matrices with elements having values anywhere in the closed interval [0, 1]) to define finite fuzzy relations.

When the related universes X and Y of a fuzzy relation R are finite such that $|X| = m, |Y| = n$, a fuzzy matrix $R = [r_{ij}]_{m\times n}$ whose generic term $r_{ij} = \mu_R(x_i, y_j)$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ where the function $\mu_R : X \times Y \to [0, 1]$ is called the membership function and r_{ij} is the grade of membership of the element (x_i, y_j) in R.

Definition 2.1 [8,9]**.** Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times l}$ be two fuzzy matrices. Then the *max–min* composition ($\circ$) of A and B is denoted by $A \circ B$ and is defined as

$$A \circ B = [t_{ij}]_{m\times l} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj}).$$

The *min–max* composition ($*$) of A and B is denoted by $A^* B$ and is defined as

$$A * B = [s_{ij}]_{m\times l} = \bigwedge_{k=1}^{n} (a_{ik} \vee b_{kj}),$$

where $\vee, \wedge$ are the maximum and minimum operations respectively.

Definition 2.2 (bifuzzy matrix [6,10,11])**.** Let $A' = [a'_{ij}]_{m\times n}, A'' = [a''_{ij}]_{m\times n}$ be two fuzzy matrices such that $a'_{ij} + a''_{ij} \le 1$ for every $i \le m, j \le n$. The pair (A', A'') is called a bifuzzy matrix and we may write $A = [a_{ij} = \langle a'_{ij}, a''_{ij}\rangle]_{m\times n}$. The numbers a'_{ij} and a''_{ij} denote the degree of membership and the degree of non-membership of the ij^{th} element in A respectively. Thus the bifuzzy matrix A takes its elements from the set $F = \{< a', a'' >: a', a'' \in [0, 1], a' + a'' \le 1\}$

For each bifuzzy matrix A of kind $m \times n$, there is a fuzzy matrix π_A associated with A such that $\pi_{ij} = 1 - a'_{ij} - a''_{ij}$ for every $i \le m, j \le n$. The number π_{ij} is called the degree of indeterminacy of the ij^{th} element in A or called the degree of hesitancy of ij^{th} element in A. It is obvious that $0 \le \pi_{ij} \le 1$ for every $i \le m, j \le n$. Especially, if $\pi_{ij} = 0$ for all $i \le m, j \le n$, then the bifuzzy matrix A is reduced to the ordinary fuzzy matrix. Thus fuzzy matrices are special cases from bifuzzy matrices.

Now, we define some operations on the set F defined above.
For $a =< a', a'' >, b =< b', b'' >\in F$, we define:
$a \wedge b =< min(a', b'), max(a'', b'') >$,
$a \vee b =< max(a', b'), min(a'', b'') >$,
$a^c =< a'', a' >$ and $a \le b$ if and only if $a' \le b', a'' \ge b''$,

$$a \ominus b = \begin{cases} < 0, a'' > & \text{if } a' \le b', a'' < b'', \\ < 0, 1 > & \text{if } a' \le b', a'' \ge b'', \\ < a', a'' > & \text{if } a' > b'. \end{cases}$$

We may write $\mathbf{0}$ instead of $< 0, 1 >$ and $\mathbf{1}$ instead of $<1, 0>$.

For the bifuzzy matrices $A = [a_{ij} = \langle a'_{ij}, a''_{ij}\rangle]_{n\times n}$, $B = [b_{ij} = \langle b'_{ij}, b''_{ij}\rangle]_{n\times n}$ and $C = [c_{ij} = \langle c'_{ij}, c''_{ij}\rangle]_{n\times m}$, let us define the following matrix operations [8–11].

$A \wedge B = [a_{ij} \wedge b_{ij}]$,

$A \vee B = [a_{ij} \vee b_{ij}]$,

$A \ominus B = [a_{ij} \ominus b_{ij}]$,

$$A * C = \left[\left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee c'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge c''_{kj})\right\rangle\right],$$

$$A \circ C = \left[\left\langle \bigvee_{k=1}^{n} (a'_{ik} \wedge c'_{kj}), \bigwedge_{k=1}^{n} (a''_{ik} \vee c''_{kj})\right\rangle\right].$$

For simplictly write AC instead of $A\circ C$. However,
$A^k = A^{k-1}A$, where
$A^k = \left[a_{ij}^{(k)} = \langle a'^{(k)}_{ij}, a''^{(k)}_{ij}\rangle\right] = A^{k-1}A$ and
$I_n = A^0 = \begin{cases} \mathbf{1} \text{ if } i = j, \\ \mathbf{0} \text{ if } i \ne j. \end{cases}$,
$A^t = [a_{ji}]$ (the transpose of A),
$A^c = [a_{ji} = \langle a''_{ij}, a'_{ij}\rangle]$ (the complement of A),
$A \le B$ if and only if $a_{ij} \le b_{ij}$ for every $i, j. \le n$.

3. Theoretical results of the paper

Definition 3.1 (reflexive, irreflexive bifuzzy matrix [6,8,9,11])**.** An $n \times n$ bifuzzy matrix $A = [a_{ij}]$ is called reflexive (irreflexive) if and only if $a_{ii} = \mathbf{1}$ ($a_{ii} = \mathbf{0}$). It is also called weakly reflexive (nearly irreflexive) if and only if $a_{ii} \ge a_{ij}$ ($a_{ii} \le a_{ij}$) for every $i, j \le n$.

Lemma 3.2. *Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ be two nearly irreflexive bifuzzy matrices. Then $A * B \le A\vee B$.*

Proof. Let $R = A * B$ and $T = A \vee B$. Then
$r_{ij} = \left\langle \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}), \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj})\right\rangle$ and $t_{ij} =< a'_{ij} \vee b'_{ij}, a''_{ij} \wedge b''_{ij} >$. Now,

$$r'_{ij} = \bigwedge_{k=1}^{n} (a'_{ik} \vee b'_{kj}) \le a'_{ii} \vee b'_{ij} \le a'_{ij} \vee b'_{ij} = t'_{ij} \text{ and}$$

$$r''_{ij} = \bigvee_{k=1}^{n} (a''_{ik} \wedge b''_{kj}) \ge a''_{ii} \wedge b''_{ij} \ge a''_{ij} \wedge b''_{ij} = t''_{ij}.$$ Thus, we have $r_{ij} \le t_{ij}$ and so $A * B \le A\vee B$. □

It is noted that $A \vee B = B$ for $A \le B$.

Lemma 3.3. *Let A and B be two nearly irreflexive bifuzzy matrices and $A \le B$. Then $A * B \le B$.*

Proof. By Lemma 3.2. □

Definition 3.4 (symmetric, asymmetric bifuzzy matrix [6,9])**.** An $n \times n$ bifuzzy matrix $A = [a_{ij}]$ is called symmetric if and only if $A = A^t$ and it is also called asymmetric if and only if $a_{ij} \wedge a_{ji} = \mathbf{0}$ for every $i, j \le n$.

Remark. It should be noted that any asymmetric bifuzzy matrix is also irreflexive.

Proposition 3.5. *Let $A = [a_{ij=}\langle a'_{ij}, a''_{ij}\rangle]_{n\times n}$ be a symmetric and nearly irreflexive bifuzzy matrix. Then we have:*

(1) $A * A \le A$,
(2) *$A * A$ is symmetric and nearly irreflexive,*
(3) *A^2 is weakly reflexive.*

Proof. (1) By Lemmas 3.2 and 3.3.

(2) Suppose $S = A * A$. It is obvious that S is symmetric and so

$$s'_{ii} = \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{ki}) = \bigwedge_{k=1}^{n} a'_{ik} \le \bigwedge_{k=1}^{n} (a'_{ik} \vee a'_{kj}) = s'_{ij}$$

and

$$s''_{ii} = \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{ki}) = \bigvee_{k=1}^{n} a''_{ik.} \ge \bigvee_{k=1}^{n} (a''_{ik} \wedge a''_{kj}) = s''_{ij}.$$

Thus, $s_{ii} \le s_{ij}$ and so that$S$ is nearly irreflexive.

(3) Let $T = A^2$. Then

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