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Closed form expression of tractable semi-max-type two-dimensional system of difference equations with variable coefficients

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Abstract In this paper we shall examine the periodicity and formularization of the solutions for a system of semi-max-type difference equations of second order in the form

$$x_{n+1} = \max\left\{\frac{A_n}{y_{n-1}}, x_{n-1}\right\},$$
$$y_{n+1} = \min\left\{\frac{B_n}{x_{n-1}}, y_{n-1}\right\},$$

$n \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are two-periodic positive sequences, and initial values $x_0, x_{-1}, y_0, y_{-1} \in (0, +\infty)$.

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1. Introduction

In recent years, the studying of nonlinear difference equations has been a considerable solicitude where there exist abundant models portray sociology, economics, real life situations in population biology, genetics, probability theory, psychology etc. whose exemplified by these kinds of equations (see, e.g. [1–8]). Also some papers are devoted to the implementing of max-type difference equations, see, for proverb [9] and references cited therein. In particular many experts have been focused on the investigation of the behavior of the following difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, x_{n-s}\right\}, \quad n \in \mathbb{N}_0, \qquad (1)$$

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where $s, k \in \mathbb{N}$, and $(A_n)_{n\in\mathbb{N}_0}$ is a sequence of real numbers (see, for example, [10,11]). Positive solutions of Eq. (1) are usually related with the periodicity. If solutions are not of constant sign then it is known that Eq. (1) can have non-periodic solutions which could be even unbounded [12]. For more papers about max-type difference equations we refer the reader to the references [13–16]. Motivated by above mentioned papers, here we will study solutions of the system of difference equations

$$x_{n+1} = \max\left\{\frac{A_n}{y_{n-1}}, x_{n-1}\right\},$$

$$y_{n+1} = \min\left\{\frac{B_n}{x_{n-1}}, y_{n-1}\right\} \qquad (2)$$

where $n \in \mathbb{N}_0$, $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are positive periodic sequences with period two, and initial values x_0, x_{-1}, y_0, $y_{-1} \in (0, +\infty)$.

2. Closed form expression of system (2)

In the section, we study the behavior of solutions of system (2). In order to achieve this target , we sub-edit and establish four theorems depending on the relationships between the quantities $\frac{A_0}{y_{-1}}$ and x_{-1}, and, $\frac{B_0}{x_{-1}}$ and y_{-1}.

Remark 1. From (2), we can note that every even (respectively odd) term of the sequences (x_n), (y_n) depend only on A_0, B_0 (respectively A_1, B_1) and the previous even (respectively odd) terms of both (x_n), (y_n).

Definition 1. A solution $(x_n, y_n)_{n=-1}^{\infty}$ of system (2), is said to be *eventually periodic with period* $p \in \mathbb{N}$ if there is an $n_0 \geq -1$, such that $x_{n+p} = x_n$, $y_{n+p} = y_n$ for $n \geq n_0$. If $n_0 = -1$, then we say that the sequence $(x_n, y_n)_{n=-1}^{\infty}$ is *periodical and with period p*. Period p is said to be a *minimal* one if there is no $p_1 < p$ which is a period to the sequence $(x_n, y_n)_{n=-1}^{\infty}$.

Theorem 2. *Suppose that (x_n, y_n) is a solution of system (2) such that $\frac{A_0}{y_{-1}} \leq x_{-1}$, $\frac{B_0}{x_{-1}} \leq y_{-1}$. Then the following statements hold:*

1. *If $y_0 \leq \frac{B_1}{x_0}$, $x_0 \geq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then*
   $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$;$x_{4n+2} = x_{4n+4} = x_0$; $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$;
   $y_{4n+1} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$; $y_{4n+2} = y_{4n+4} = y_0$; $y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$
   (b) *If $1 \geq \frac{A_0}{B_0}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$;$x_{4n+2} = x_{4n+4} = x_0$;
   $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$; $y_{4n+2} = y_{4n+4} = y_0$
2. *If $y_0 \geq \frac{B_1}{x_0}$, $x_0 \leq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$; $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$; $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$; $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$; $y_{4n+1} = y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$;
   $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$; $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$;
   (b) *If $1 \geq \frac{A_0}{B_0}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = \frac{A_1^{n+1}}{B_1^n y_0}$,
   $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = \frac{B_1^{n+1}}{A_1^n x_0}$,
   $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
3. *If $y_0 \leq \frac{B_1}{x_0}$, $x_0 \leq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then*
   i. *If $1 \leq \frac{B_1}{A_1}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2} = x_{4n+4} = \frac{A_1}{y_0}$,
   $x_{4n+3}=\frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$, $y_{4n+1}=y_{4n+3}=\frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2}=y_{4n+4}=y_0$,
   ii. *If $1 \geq \frac{B_1}{A_1}$, then*
   $x_{4n+1}=\frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2}=x_{4n+4} = \frac{A_1^{n+1}}{B_1^n y_0}$, $x_{4n+3}=\frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$,
   $y_{4n+1}=y_{4n+3}=\frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = \frac{B_1^n}{A_1^n}y_0$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
   (b) *If $1 \geq \frac{A_0}{B_0}$, then*
   i. *If $1 \leq \frac{B_1}{A_1}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4} = \frac{A_1}{y_0}$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = y_{4n+4} = y_0$,
   ii. *If $1 \geq \frac{B_1}{A_1}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = x_{4n+4} = \frac{A_1^{n+1}}{B_1^n y_0}$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = \frac{B_1^n}{A_1^n}y_0$, $y_{4n+4} = \frac{B_1^{n+1}}{A_1^{n+1}}y_0$
4. *If $y_0 \geq \frac{B_1}{x_0}$, $x_0 \geq \frac{A_1}{y_0}$, then*
   (a) *If $1 \leq \frac{A_0}{B_0}$, then*
   i. *If $1 \leq \frac{A_1}{B_1}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2} = \frac{A_1^n}{B_1^n}x_0$, $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$, $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$,
   $y_{4n+2} = y_{4n+4} = \frac{B_1^{n+1}}{A_1^n x_0}$,
   ii. *If $1 \geq \frac{A_1}{B_1}$, then* $x_{4n+1} = \frac{A_0^n}{B_0^n}x_{-1}$, $x_{4n+2} = x_{4n+4} = x_0$,
   $x_{4n+3} = \frac{A_0^{n+1}}{B_0^{n+1}}x_{-1}$,$y_{4n+1} = y_{4n+3} = \frac{B_0^{n+1}}{A_0^n x_{-1}}$, $y_{4n+2} = y_{4n+4} = \frac{B_1}{x_0}$
   (b) *If $1 \geq \frac{A_0}{B_0}$, then*
   i. *If $1 \leq \frac{A_1}{B_1}$, then* $x_{4n+1} = x_{4n+3} = x_{-1}$, $x_{4n+2} = \frac{A_1^n}{B_1^n}x_0$,
   $x_{4n+4} = \frac{A_1^{n+1}}{B_1^{n+1}}x_0$, $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$, $y_{4n+2} = y_{4n+4} = \frac{B_1^{n+1}}{A_1^n x_0}$
   ii. *If $1 \geq \frac{A_1}{B_1}$, then*
   $x_{4n+1} = x_{4n+3} = x_{-1}$,$x_{4n+2} = x_{4n+4} = x_0$,
   $y_{4n+1} = y_{4n+3} = \frac{B_0}{x_{-1}}$,$y_{4n+2} = y_{4n+4} = \frac{B_1}{x_0}$

Proof. By mathematical induction. For $n = 0$, the result holds. Now suppose that $k > 0$ and that all the relations in the theorem hold for $n = k$. Now we shall prove that the relations hold for $n = k + 1$.

1. If $y_0 \leq \frac{B_1}{x_0}$, $x_0 \geq \frac{A_1}{y_0}$, then
   (a) If $1 \leq \frac{A_0}{B_0}$, then
   $x_{4(k+1)+1} = \max\{\frac{A_{4(k+1)}}{y_{4(k+1)-1}}, x_{4(k+1)-1}\} = \max\{\frac{A_0}{y_{4k+3}}, x_{4k+3}\}$
   $= \max\{\frac{A_0}{\frac{B_0^{k+1}}{A_0^k x_{-1}}}, \frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}\} = \frac{A_0^{(k+1)}}{B_0^{(k+1)}}x_{-1}$
   $x_{4(k+1)+2} = \max\{\frac{A_{4k+5}}{y_{4k+4}}, x_{4k+4}\} = \max\{\frac{A_1}{y_0}, x_0\} = x_0$
   $y_{4(k+1)+1} = \min\{\frac{B_{4(k+1)}}{x_{4(k+1)-1}}, y_{4(k+1)-1}\} =$
   $\min\{\frac{B_0}{\frac{A_0^{k+1}}{B_0^{k+1}}x_{-1}}, \frac{B_0^{k+1}}{A_0^k x_{-1}}\} = \frac{B_0^{k+2}}{A_0^{k+1}x_{-1}}$
   $y_{4(k+1)+2} = \min\{\frac{B_{4k+5}}{x_{4k+4}}, y_{4k+4}\} = \min\{\frac{B_1}{x_0}, y_0\} = y_0$

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