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Topological representation and quantic separation axioms of semi-quantales

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Abstract An adjunction between the category of semi-quantales and the category of lattice-valued quasi-topological spaces is established. Some characterizations of quantic separation axioms, for semi-quantales and lattice-valued quasi-topological spaces, are obtained and some relations among these axioms are established.

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1. Introduction

Quantales were first introduced in the eighties by Mulvey [1] in the ambitious aim of providing a possible common lattice-theoretic setting for constructive foundations for quantum mechanics, as well as a non-commutative analogue of the maximal spectrum of a C^*-algebra, and for non-commutative logics. The study of such ordered algebraic structures goes back to a series of papers by Ward and Dilworth [2–4] in the 1930s. They were motivated by the ideal theory of commutative rings. Following Mulvey, various types and aspects of quantales have been considered by many authors [5–8].

Since quantale theory provides a powerful tool in studying non-commutative structures, it has a wide applications, especially in studying non-commutative C^*-algebra theory [6,9], the ideal theory of commutative ring [10], linear logic [11] which supports part of the foundation of theoretic computer science [12,13] and so on.

In 1989 Borceux and van den Bossche [14] proposed a duality between spatial right-sided idempotent quantales and sober quantum spaces. In 2015, Höhle [15] established two adjunctions based on right-sided idempotent quantales. The first adjunction based on quantum spaces as an extension of the duality between spatial right-sided idempotent quantales and sober quantum spaces. The second adjunction between the category of right-sided idempotent quantales and the category of three-valued topological spaces. Both adjunctions restricts to the well known Papert–Papert–Isbell adjunction [16,17] between topological spaces and locales. In 2014 Demirci [18] established an

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abstract categorical analogue of famous Papert–Papert–Isbell adjunction to a general adjunction $X \dashv C^{op}$ in which C is an abstract category and X is a suitable category of such counterparts. Also he formulated two main categorical theorems: Fundamental Categorical Adjunction Theorem (FCAT) and Fundamental Categorical Duality Theorem (FCDT).

In this paper we aim to introduce and study a more general adjunction between the category of semi-quantales [19] and the category of lattice-valued quasi-topological spaces [20]. Also, we aim to study some separation axioms for semi-quantales with applications to lattice-valued quasi-topological spaces.

The present paper has been prepared in four sections. After this introductory section, the next section overviews the some useful concepts about semi-quantales, quantic nucleus and L-quasi-topologies. In Section 3, as one of the main contribution of this paper, we construct a dual adjunction between the category **SQuant** of semi-quantales and the category ***L*-QTop** of lattice-valued quasi-topological spaces. Also, by defining L-Qspatiality in the given category **SQuant** and L-Qsobriety in ***L*-QTop**, we show that the full subcategory of **SQuant** of all L-Qspatial objects and the full subcategory of ***L*-QTop** of all L-Qsober objects are dually equivalent. The results of this section can be obtained as applications of Fundamental Categorical Adjunction Theorem (FCAT) and Fundamental Categorical Duality Theorem (FCDT) [18]. Finally in Section 4, we will discuss the counterparts of the quantic regularity and normality axioms of objects in the category **SQuant** with applications to objects in the category *L*-**QTop**.

2. Preliminaries

By a $\bigvee$-semilattice we mean a partially ordered set $(L, \leq)$ having arbitrary $\bigvee$. A $\bigvee$-semilattice homomorphism is a map preserving arbitrary $\bigvee$.

Definition 2.1 ([19]). (lattice structures and associated categories).

(1) A semi-quantale $(L, \leq, \otimes)$, abbreviated as s-quantale, is a $\bigvee$-semilattice $(L, \leq)$ equipped with a binary operation $\otimes : L \times L \longrightarrow L$, with no additional assumptions, called a tensor product. The category **SQuant** comprises all semi-quantales together with s-quantale morphisms (i.e., mappings preserving $\otimes$ and arbitrary $\bigvee$). By **SSQuant** [20], we mean a non-full subcategory of **SQuant** comprising all semi-quantales and all ss-quantale morphisms (i.e., mappings preserving $\otimes$, arbitrary $\bigvee$ and $\top$). **SSQuant** and **SQuant** clearly share the same objects.
(2) A quantale $(L, \leq, \otimes)$ is an s-quantale whose multiplication is associative and distributes across $\bigvee$ from both sides [7]. **Quant** is the full subcategory of **SQuant** of all quantales.
(3) An ordered semi-quantale $(L, \leq, \otimes)$, abbreviated as os-quantale, is an s-quantale in which $\otimes$ is isotone in both variables. **OSQuant** is the full subcategory of **SQuant** of all os-quantales.
(4) A unital semi-quantale $(L, \leq, \otimes)$, abbreviated as us-quantale, is an s-quantale in which $\otimes$ has an identity element $e \in L$ called the unit. **USQuant** comprises all us-quantales together with all mappings preserving arbitrary $\bigvee$, $\otimes$, and e.
(5) A commutative semi-quantale $(L, \leq, \otimes)$, abbreviated as cs-quantale, is an s-quantale in which, $\otimes$ that is, $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in L$. **CSQuant** is the full subcategory of **SQuant** of all commutative semi-quantales.
(6) A complete quasi-monoidal lattice $(L, \leq, \otimes)$, abbreviated as cqml, is an os-quantale having $\top$ idempotent i.e., $\top \otimes \top = \top$. **CQML** comprises all cqml together with mappings preserving arbitrary $\bigvee$, $\otimes$, and $\top$ [21,22]. Note that **CQML** is a subcategory of **OSQuant**.
(7) A semi-frame [22] is a us-quantale whose multiplication and unit are $\wedge$ and $\top$ respectively. **SFrm** is the category of all semi-frames together with mappings preserving finite $\wedge$ and arbitrary $\bigvee$. **SFrm** is a full subcategory of **CQML**.
(8) A frame [23] is a unital quantale whose multiplication and unit are $\wedge$ and $\top$ respectively. **Frm** is the subcategory of **Quant** of all frames and morphisms preserving finite $\wedge$ and arbitrary $\bigvee$.

Definition 2.2 ([24]). An s-quantale is called distributive (ds-quantale) provided that its multiplication distributes across finite $\vee$ from both sides. **DSQuant** is the category of ds-quantales.

Definition 2.3 ([20]). Let $L = (L, \leq, \otimes)$ be an s-quantale. A subset $K \subseteq L$ is a subsemi-quantale of L iff it is closed under the tensor product $\otimes$ and arbitrary $\bigvee$. A subsemi-quantale K of L is said to be strong iff $\top$ belongs to K. If L is a us-quantale with the identity e, then a subsemi-quantale K of L is called a unital subsemi-quantale of L iff e belongs to K.

Definition 2.4 ([25]). Let Q be a semi-quantale. An element $\top \neq p \in Q$ is said to be prime if $a \otimes b \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in Q$. The set of all prime elements of Q, denoted by $Pr(Q)$.

Definition 2.5 (see [7]). Let $Q \in |\mathbf{SQuant}|$. A quantic nucleus on Q is a closure operator j: $Q \to Q$ such that $j(a) \otimes j(b) \leq j(a \otimes b)$ for all $a, b \in Q$.

A subset $S \subseteq Q$ is called a quantic quotient if $S = Q_j$ for some quantic nucleus j, where $Q_j = \{a \in Q : j(a) = a\}$.

Let X be a non-empty set and let L be a complete lattice or $L \in |\mathbf{SQuant}|$. An L-fuzzy subset (or L-set) of X is a mapping A: $X \to L$. The family of all L-fuzzy subsets on X will be denoted by L^X. The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively.

For an ordinary mapping $f : X \longrightarrow Y$, one can define the mappings

$$f_L^{\rightarrow} : L^X \to L^Y \text{and } f_L^{\leftarrow} : L^Y \to L^X$$

by

$$f_L^{\rightarrow}(A)(y) = \bigvee \{A(x) : x \in X, f(x) = y\} \text{and } f_L^{\leftarrow}(B) = B \circ f$$

respectively.

Theorem 2.6 ([19]). *Let $L \in |\mathbf{SQuant}|$, X, Y be a nonempty ordinary sets and $f : X \longrightarrow Y$ be an ordinary mapping, then we have:*

(1) *$f_L^{\rightarrow}$ preserves arbitrary $\bigvee$;*
(2) *$f_L^{\leftarrow}$ preserves arbitrary $\bigvee$, $\otimes$, and all constant maps;*
(4) *$f_L^{\leftarrow}$ preserves the unit if $L \in |\mathbf{USQuant}|$.*

For a fixed $L \in |\mathbf{SQuant}|$ and a set X, an L-quasi-topology on X [19] is a subs-quantale τ of $L^X = (L^X, \leq, \otimes)$, i.e., the following axioms are satisfied: