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Original Article

On retracting properties and covering homotopy theorem for S-maps into S_χ-cofibrations and S_χ-fibrations

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Abstract In this paper we generalize the retracting property in homotopy theory for topological semigroups by introducing the notions of deformation S-retraction with its weaker forms and ES-homotopy extension property. Furthermore, the covering homotopy theorems for S-maps into S_χ-fibrations and S_χ-cofibrations are introduced and pullbacks for S_χ-fibrations behave properly.

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1. Introduction

The homotopy theory is an important part of mathematics which has many applications and numerous variants, generalizations, and adaptations. It has been improved to the shape theory in order to deal better with spaces with poor local properties. The concepts of Hurewicz fibrations [3] and retractions [1] have p layed very important roles for investigating the mutual relations among the topological spaces.

Under the notion of homotopy theory for topological spaces, Cerin in [2] introduced the definition of homotopy theory for topological semigroups. He extended some basic properties in homotopy theory to their analogous structures in homotopy theory for topological semigroups such as S-retraction, K-retraction, S-homotopically domination, S_χ-fibration and S_χ-cofibration.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we give the concepts of deformation S-retract, deformation K-retract, strong deformation S-retraction, and ES-homotopy extension property. The S_χ-fibrations and S_χ-cofibrations played very important roles for investigating the mutual relations of among these concepts. In

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Section 4 we introduce the covering homotopy theorems for S-maps into S_χ-fibrations and S_χ-cofibrations. We prove the pullbacks for S_χ-fibrations are S_χ-fibrations.

2. Preliminaries

In this section we provide some preliminary works that serve as background for the present study which were previously established by Cerin, in [2].

A *topological semigroup* or *S-space* is a pair (S, a) consisting a topological space S and a map (i.e., a continuous function) a: $S \times S \to S$ such that $a(x, a(y, z)) = a(a(x, y), z)$ for all $x, y, z \in S$. Let χ denotes the class of all S-spaces.

For every space S, the *natural S-space* is S-space (S, π_i), where π_i is a continuous associative multiplication on S given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $x, y \in S$. We denote the class of all natural S-spaces (S, π) by $\mathcal{N}_\pi$, where $\pi = \pi_1, \pi_2$.

S-space (B, c) is called an *S-subspace* of (S, a) if B is a subspace of S and the map a takes the product $B \times B$ into B and $c(x, y) = a(x, y)$ for all $x, y \in B$. It is natural to denote the multiplication of S-subspace with the same symbol used for the multiplication on the S-space under consideration.

Let (S, a) and (O, e) be two S-spaces. The function f: $(S, a) \to (O, e)$ is called a *homomorphism* or *an S-map* if f is a map of a space S into O and $f(a(x, y)) = e(f(x), f(y))$ for all $x, y \in S$. Recall [2] that the usual composition and the usual product of two S-maps are S-maps.

For every a space S, by $P(S)$ we mean the space of all paths from the unit closed interval $I = [0, 1]$ into S with the compact-open topology. Recall [2] that for every S-space (S, a), $(P(S), \underline{a})$ is S-space where $\underline{a} : P(S) \times P(S) \to P(S)$ is a map defined by $\underline{a}(\alpha, \beta)(t) = a(\alpha(t), \beta(t))$ for all $\alpha, \beta \in P(S)$, $t \in I$. The shorter notion for this S-space will be $P(S, a)$.

Definition 2.1. The S-maps f, g: $(S, a) \to (O, e)$ are called S-homotopic and write $f \simeq {}_s g$ provided there is S-map H: $(S, a) \to P(O, e)$ called S-homotopy such that $H(s)(0) = f(s)$ and $H(s)(1) = g(s)$ for all $s \in S$.

Theorem 2.2. *The relation of S-homotopy $\simeq_s$ is an equivalence relation on the set of all S-maps of (S, a) into (O, e).*

Theorem 2.3. *If the S-maps f, g: $(S, a) \to (O, e)$ are S-homotopic then the relations $f \circ h \simeq {}_s g \circ h$ and $k \circ f \simeq {}_s k \circ g$ hold for all S-maps h into (S, a) and k from (O, e).*

Recall [2] that if the S-maps f, g: $(S, a) \to (O, e)$ are S-homotopic then the maps f, g: $S \to O$ are homotopic and the S-maps f, g: $(S, \pi) \to (O, \pi)$ are S-homotopic if and only if the maps f, g: $S \to O$ are homotopic.

Throughout this paper, for every S-homotopy H: $(S, a) \to P(O, e)$ and for every $t \in I$, by H_t (or $[H]_t$) we mean the S-map, [2], H_t: $(S, a) \to (O, e)$ which given by $H_t(s) = H(s)(t)$ for all $s \in S$. Also for every S-homotopy H: $(S, a) \to P[P(O), \underline{e}]$ and for every $r, t \in I$, by H_{rt} (or $[H]_{rt}$) we mean the S-map H_{rt}: $(S, a) \to (O, e)$ which given by $H_{rt}(s) = [H(s)(r)](t)$ for all $s \in S$.

Definition 2.4. S-map f: $(S, a) \to (O, e)$ is called S_χ-fibration if for every space $(X, c) \in \chi$, S-map g: $(X, c) \to (S, a)$, and S-homotopy G: $(X, c) \to P(O, e)$ with $G_0 = f \circ g$, there is S-homotopy H: $(X, c) \to P(S, a)$ such that $H_0 = g$ and $f \circ H_t = G_t$ for all $t \in I$.

Recall [2] that the map f: $S \to O$ is a Hurewicz fibration if and only if the S-map f: $(S, \pi) \to (O, \pi)$ is $S_{\mathcal{N}_\pi}$−fibration.

Definition 2.5. *S-map f: $(S, a) \to (O, e)$ is called* S_χ-cofibration*if for every space $(X, c) \in \chi$, S-map g: $(O, e) \to (X, c)$, and S-homotopy G: $(S, a) \to P(X, c)$ with $G_0 = g \circ f$, there is S-homotopy H: $(O, e) \to P(X, c)$ such that $H_0 = g$ and $H_t \circ f = G_t$ for all $t \in I$.*

Recall [2] that the map f: $S \to O$ is a cofibration if and only if the S-map f: $(S, \pi) \to (O, \pi)$ is $S_{\mathcal{N}_\pi}$−cofibration.

Definition 2.6. *An S-subspace (B, a) of S-space (S, a) is called* S-retract*of (S, a) if there exists S-map R: $(S, a) \to (B, a)$ such that $R(s) = s$ for all $s \in B$. The S-map R is called* S-retraction*of (S, a) onto (B, a).*

Throughout this paper, j: $(B, a) \to (S, a)$ will denote to the inclusion S-map for every S-subspace (B, a) of S-space (S, a) and *id* the identity S-map.

Definition 2.7. An S-subspace (B, a) of S-space (S, a) is called K-retract of (S, a) if there exists S-map r: $(S, a) \to (B, a)$ such that $r \circ j \simeq {}_s id_B$. The S-map r is called K-retraction of (S, a) onto (B, a).

Notice that S-retract is an K-retract. The converse of the first claim is not true in general. In the following theorem, [2] proved a sufficient condition.

Theorem 2.8. *Let (B, a) be S-subspace of S-space (S, a) such that the inclusion S-map j: $(B, a) \to (S, a)$ is $S_{\{(B, a)\}}$-cofibration. Then (B, a) is S-retract of (S, a) if and only if (B, a) is K-retract of (S, a).*

3. Deformation S-retractions

Definition 3.1. An S-subspace (B, a) of S-space (S, a) is called a *deformation S-retract* of (S, a) if there exists S-retraction map R: $(S, a) \to (B, a)$ of (S, a) onto (B, a) such that $j \circ R \simeq {}_s id_S$. The S-homotopy between $j \circ R$ and id_S is called a deformation S-retraction.

Example 3.2. Let (S, a) be S-space and $s_o \in S$ be an idempotent element of (S, a) (i.e., $s_o a s_o = s_o$). Let

$$L(S, s_o) = \{\alpha \in P(S) : \alpha(0) = s_o\} \subset P(S)$$

and $\widetilde{s_o}$ be the constant path at s_o in $L(S, s_o)$. For every $\alpha, \beta \in L(S, s_o)$,

$$(\alpha \underline{a} \beta)(0) = \alpha(0) a \beta(0) = s_o a s_o = s_o.$$

That is, a pair $(L(S, s_o), \underline{a})$ is S-subspace of $P(S, a)$. Similarly, $(\{\widetilde{s_o}\}, \underline{a})$ is S-subspace of $(L(S, s_o), \underline{a})$. Define the S-retraction $R : (L(S, s_o), \underline{a}) \to (\{\widetilde{s_o}\}, \underline{a})$ by $F(\alpha) = \widetilde{s_o}$ for all $\alpha \in L(S, s_o)$. $(\{\widetilde{s_o}\}, \underline{a})$ is a deformation S-retract of $(L(S, s_o), \underline{a})$ such that $id_{L(S, s_o)} \simeq_s j \circ R$ by a deformation S-retraction F: $(L(S, s_o), \underline{a}) \to P(L(S, s_o), \underline{a})$ given by $F_{rt}(\alpha) = \alpha(r(1 - t))$ for all $r, t \in I$, $\alpha \in L(S, s_o)$, where $j : (\{\widetilde{s_o}\}, \underline{a}) \to (L(S, s_o), \underline{a})$ is the inclusion S-map.

The S-map f: $(S, a) \to (O, e)$ is called *S-homotopy equivalence* if there exists S-map g: $(O, e) \to (S, a)$ such that $f \circ g \simeq {}_s id_O$ and