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Strong semilattices of topological groups

A.M. Abd-Allah, A.I. Aggour*, A. Fathy

Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City Cairo 11884, Egypt

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Abstract The notion of partial groups and their basic properties have been given in [1,2]. In this paper, we introduce the concept of topological partial groups and discuss some of their basic properties. So, the category of topological partial groups Tpg, as objects, and the homomorphisms of topological partial groups, as arrows, have some deficiencies. To get over these deficiencies, we introduced the category of locally compact partial groups denoted by Lcpg. Finally, we introduced the category of strong semilattices of topological groups denoted by Sstg.

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1. Preliminaries

We collect for sake of reference the needed definitions and results appeared in the given references.

Definition 1.1 ([3,4]). A topological group G is a pair (G, τ), where G is a group and τ is a topology on G which satisfies the continuity of the following maps:

(i) $\mu: G \times G \to G; (x, y) \mapsto xy$;
(ii) $\gamma: G \to G; x \mapsto x^{-1}$.

Theorem 1.1 [3]. *If G is a topological group, then γ is a homoeomorphism.*

Theorem 1.2 [4]. *A group G with a topology τ is a topological group if and only if the map f: $G \times G \to G$, $(x, y) \mapsto x^{-1}y$ is continuous.*

Definition 1.2 [3]. Let G and H be topological groups, then ϕ: $G \to H$ is called a morphism if ϕ is continuous and a group homomorphism.

Definition 1.3 [4]. Let G be a topological group and B be a subgroup of G. Then B with the relative topology is called a topological subgroup.

Theorem 1.3 [4]. *B is a topological subgroup of a topological group G if and only if the inclusion map i: $B \to G$ is a morphism.*

Definition 1.4 [5]. Let S be a semigroup. Then $x \in S$ is called an idempotent element if $x \cdot x = x$. The set of all idempotent elements in S is denoted by $E(S)$.

* Corresponding author. Tel.: +20 2 22629009; fax: +20 2 22629009.
E-mail address: atifaggour@yahoo.com (A.I. Aggour).
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Definition 1.5 [2]. Let S be a semigroup and $x \in S$. Then $e \in S$ is called a partial identity of x if

(i) $ex = xe = x$;
(ii) If $e'x = xe' = x$, $e' \in S$, then $ee' = e'e = e$.

Theorem 1.4 [2]. *If S is a semigroup, then*

(i) *If $x \in S$ has a partial identity, then it is unique.*
(ii) *$E(S)$ is the set of all partial identities of the elements of S.*

We will denote by e_x the partial identity of the element $x \in S$.

Definition 1.6 [2]. Let S be a semigroup and $x \in S$ has a partial identity e_x. The element $y \in S$ is called a partial inverse of x if

(i) $xy = yx = e_x$.
(ii) $e_x y = ye_x = y$.

Theorem 1.5 [2]. *Let S be a semigroup and $x \in S$ has a partial identity e_x. If x has a partial inverse y, then it is unique.*

We will denote by x^{-1} the partial inverse of $x \in S$.

Theorem 1.6 [2]. *Let S be a semigroup and $x \in S$. Then:*

(i) $(e_x)^{-1} = e_x$, $\forall\ e_x \in E(S)$.
(ii) $e_{x^{-1}} = e_x$.
(iii) $(x^{-1})^{-1} = x$.

Definition 1.7 [2]. A semigroup S is called a partial group if:

(i) Every $x \in S$ has a partial identity e_x.
(ii) Every $x \in S$ has a partial inverse x^{-1}.
(iii) The map e_S: $S \to S$; $x \mapsto e_x$ is a semigroup homomorphism.
(iv) The map γ: $S \to S$, $x \mapsto x^{-1}$ is a semigroup anti-homomorphism $[(xy)^{-1} = y^{-1}x^{-1}]$.

From this definition we have every group is a partial group. So, the notion of partial group is a good generalization of that of group. So, it is important to study a reasonable topology on a partial group to satisfy the nice properties of topological groups.

Definition 1.8 [2]. If S is a partial group and $x \in S$, then we define

$$S_x = \{y \in S : e_x = e_y\}.$$

Theorem 1.7 [2]. *Let S be a partial group and $x \in S$, then*

(i) *S_x is a maximal subgroup of S which has identity e_x.*
(ii) $S = \cup\{S_x : x \in S\} = \cup\{S_{e_x} : e_x \in E(S)\}$.

Corollary 1.1 [2]. *Every partial group is a disjoint union of a family of groups.*

Theorem 1.8 [2]. *Let S be a partial group, then $E(S)$ is commutative and central.*

Definition 1.9 [1]. A subsemigroup B of a partial group S is called a subpartial group, denoted by $B \leq S$, if $\forall\ x \in B$ we have $x^{-1} \in B$ and $e_x \in B$.

Theorem 1.9 [5]. *Let S be a partial group and $B \subseteq S$, then $B \leq S$ if and only if $x^{-1}y \in B$, $\forall\ x, y \in B$.*

Definition 1.10 [1]. Let S and T be partial groups, then ϕ: $S \to T$ is called a partial group homomorphism if $\phi(xy) = \phi(x)\phi(y)$, $\forall\ x, y \in S$.

Definition 1.11 [1]. Let ϕ: $S \to T$ be a partial group homomorphism, then $ker\ \phi = \{x \in S : \phi(x) = e_{\phi(x)}\}$ and $Im\ \phi = \{\phi(x) : x \in S\}$.

Definition 1.12 [1]. A partial group homomorphism ϕ: $S \to T$ is called an isomorphism if it is bijective.

Definition 1.13 [1]. If S is a partial group and $B \leq S$, then B is called normal, denoted by $B \trianglelefteq S$, if B is wide ($E(S) \subseteq B$) and $xyx^{-1} \in B$, $\forall\ x \in S$, $y \in B$.

Definition 1.14 [1]. Let S be a partial group and $B \trianglelefteq S$. The set $\{xB$: $x \in S\}$ is called the quotient set, denoted by $S|B$, where $xB = \{y \in S : x^{-1}y \in B, e_x = e_y\}$ is called the left coset of B by x .

Theorem 1.10 [1]. *Let S be a partial group and $N \trianglelefteq S$. Then $S|N$ with the map μ: $S|N \times S|N \to S|N$, $(xN, yN) \mapsto (xy)N$ is a partial group.*

Definition 1.15 [5]. Let $(S_i)_{i \in Y}$ be a family of groups indexed by a semilattice Y of the identities of the groups such that if $i \geq j$, $i, j \in Y$, there exists a group homomorphism $\phi_{i,j}$: $S_i \to S_j$, satisfies:

(i) $\phi_{i,i}$ is the identical automorphism;
(ii) $\phi_{j,k}\phi_{i,j} = \phi_{i,k}$, where $i \geq j \geq k$, $i, j, k \in Y$.

Then the disjoint union $S = \bigcup_{i \in Y} S_i$, with the binary operation $S \times S \to S$, $(x_i, y_j) \mapsto x_i y_j = (\phi_{i,ij}x_i)(\phi_{j,ij}y_j)$, $\forall\ x_i \in S_i$, $y_j \in S_j$, is called a strong semilattice of groups, denoted by $S = \mathcal{L}(S_i, Y, \phi_{i,j})$.

Theorem 1.11 [2]. *S is a partial group if and only if S is a strong semilattice of groups.*

Definition 1.16 [5]. Let ϕ: $S \to T$ be a partial group homomorphism. Then ϕ is called idempotent separating if $\phi(e_x) = \phi(e_y)$ implies that $e_x = e_y$, $\forall\ e_x, e_y \in E(S)$.

The following results are the fundamental theorems of isomorphisms.

Theorem 1.12 [1]. *Let ϕ: $S \to T$ be an idempotent separating surjective partial group homomorphism and $K = ker\ \phi$. Then there exists a unique isomorphism α: $S|K \to T$ such that $\phi = \alpha\rho_K$, where ρ_K: $S \to S|K$; $x \mapsto xK$ is the quotient map.*

Theorem 1.13 [1]. *Let $M, N \trianglelefteq S$ be such that $M \subseteq N$. Then*

(i) $N|M \trianglelefteq S|M$;
(ii) *There exists a unique isomorphism α: $(S|M)|(N|M) \to S|N$ such that $\rho_N = \alpha\rho_{N|M}\rho_M$, where ρ_N: $S \to S|N$ and $\rho_{N|M}$: $S|M \to (S|M)|(N|M)$ are the quotient maps.*

Definition 1.17 [2]. Let S be a partial group, and $A, B \subseteq S$. Then, we define $AB = \{ab : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$.

Definition 1.18 [6]. Let $X = \sqcup_{\lambda \in L} X_\lambda$ be the sum of the underlying sets of the family $(X_\lambda)_{\lambda \in L}$ of topological spaces, and let i_λ: $X_\lambda \to X$ be the inclusions. The final topology on X with respect to $(i_\lambda)_{\lambda \in L}$ is called the sum topology. Clearly, a map $f : X = \sqcup_{\lambda \in L} X_\lambda \to Y$ is continuous if and only if fi_λ is continuous, for all $\lambda \in L$.

Definition 1.19 [6]. Let $X = \sqcup_{n \in N} X_n$ and $Y = \sqcup_{m \in M} Y_m$ and $X \times Y$ be the cartesian product of X and Y. Then, $X \times Y$ with the final topology with respect to the inclusions $(i_n \times i_m)_{n \in N, m \in M}$ is called the weak product of X and Y, denoted by $X \times_W Y$.

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