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Generalized rough sets based on neighborhood systems and topological spaces

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Abstract Rough sets theory is an important method for dealing with uncertainty, fuzziness and undefined objects. In this paper, we introduce a new approach for generalized rough sets based on the neighborhood systems induced by an arbitrary binary relation. Four pairs of the dual approximation operators are generated from the core of neighborhood systems. Relationship among different approximation operators are presented. We generate different topological spaces by using the core of these neighborhood systems. Relationship among different generated topologies are discussed.

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1. Introduction

There are many mathematical tools to deal with inexact or uncertain knowledge in information systems such probability theory, fuzzy sets [1] and rough sets [2]. Rough sets was proposed by Pawlak [3] as an useful tool to deal with uncertainty and incomplete information. Since then rough sets and its applications have attracted the interest of researchers in many fields [4–17]. The indiscernibility relation is the starting point of Pawlak rough set which was first described by equivalence relation. However, the requirement of equivalence relation such as the indiscernibility impose restrictions and limitations in many applications. In the light of this, equivalence relation has been extended to some other relations such as similarity relation [18], tolerance relation [19], fuzzy relations [20], arbitrary relation [17,21–23] and coverings of the universal sets [24–31].

Topology is regarded as an important and significant branch of mathematics. In recent years many researchers have used topological approaches in the study of rough sets and its applications. The combination of topological spaces and rough sets and the properties of topological rough spaces are discussed by Wu et al. [32]. Lin [12,13,33] used neighborhood systems and topological concept in the study of approximations. Also, neighborhood systems can be induced by the binary relations. The equivalence class of each element in the equivalence relation

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can be viewed as a neighborhood of this element [34,35]. Yao [23,36] introduced the successor elements of any element in an arbitrary binary relation as its right neighborhood. A concept of neighborhood assignment of general topology is considered by Hung [8]. Zuoming et al. [30] used the same concept, is called ”core of neighborhoods”, and defined two classes of new rough sets based on neighborhood systems in terms of cores.

In this paper, we introduce a new approach for generalization rough sets based on an arbitrary binary relation via the concept of the core of neighborhoods. Four classes of new rough sets are defined. The properties of new rough sets are established and compared with the properties of other approaches. We discuss the relationship among the four approximation. We claim that our approach is an extension of the classical rough sets. We generate four different topologies in terms of cores. Relationship among four different topologies are discussed. Our paper is considered an important evidence for the relationship between topology and rough set theory.

2. Preliminaries

Definition 2.1. [2] Let U be a non empty set, is called the universe of discourse, and R be an equivalence relation on U. Then, the pair $K = (U, R)$ is called an approximation space. For any subset $X \subseteq U$, $\underline{\Re}(X)$,$\overline{\Re}(X)$ are called the lower and upper approximations, respectively, and are defined as follows:

$$\underline{\Re}(X) = \{x \in U : [x]_R \subseteq X\}, \overline{\Re}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$$

where $[x]_R$ is the equivalence class of x with respect to R.

Proposition 2.1. [2] Let $K = (U, \Re)$ be an approximation space. Then, the following properties hold, for $X, Y \subseteq U$:

(1L) $\underline{\Re}(U) = U$;
(1H) $\overline{\Re}(U) = U$;
(2L) $\underline{\Re}(\emptyset) = \emptyset$;
(2H) $\overline{\Re}(\emptyset) = \emptyset$;
(3L) $\underline{\Re}(X) \subseteq X$;.
(3H) $X \subseteq \overline{\Re}(X)$.
(4L) $\underline{\Re}(X \cap Y) = \underline{\Re}(X) \cap \underline{\Re}(Y)$;
(4H) $\overline{\Re}(X \cup Y) = \overline{\Re}(X) \cup \overline{\Re}(Y)$;
(5) $\underline{\Re}(-X) = -\overline{\Re}(X)$, where $(-X)$ is the complement of X;
(6L) $\underline{\Re}(\underline{\Re}(X)) = \underline{\Re}(X)$;
(6H) $\overline{\Re}(\overline{\Re}(X)) = \overline{\Re}(X)$;
(7L) $X \subseteq Y \Rightarrow \underline{\Re}(X) \subseteq \underline{\Re}(Y)$;
(7H) $X \subseteq Y \Rightarrow \overline{\Re}(X) \subseteq \overline{\Re}(Y)$;
(8L) $\underline{\Re}(-\underline{\Re}(X)) = -\underline{\Re}(X)$;
(8H) $\overline{\Re}(-\overline{\Re}(X)) = -\overline{\Re}(X)$;
(9L) $\underline{\Re}(X) \cup \underline{\Re}(Y) \subseteq \underline{\Re}(X \cup Y)$;
(9H) $\overline{\Re}(X \cap Y) \subseteq \overline{\Re}(X) \cap \overline{\Re}(Y)$;

Definition 2.2. [36] Let R be a binary relation on the universe U and $x, y \in U$. If $(x, y) \in R$, then we say that y is related to x by R and the class $RN(x) = \{y \in U : xRy\}(LN(x) = \{y \in U : yRx\})$ is called the right neighbored (the left neighbored) of x induced by R, respectively.

Definition 2.3. [8,30] Let R be a binary relation on the universe U and $x, y \in U$. Then, the set $\{y \in U : N(y) = N(x)\}$ is called the core of neighborhood of x induced by R and is denoted by $CN(x)$.

Definition 2.4. [37] Let U be a non empty set, τ be a family of subsets of U and the following properties hold:

(i) $U, \emptyset \in \tau$;
(ii) τ is closed under an arbitrary union;
(iii) τ is closed under finite intersection.

Then, τ is called a topology on U and the pair (U, τ) is called a topological space. The elements of U are called points of the space. The subsets of U belonging to τ are called open sets and the complement of the open subsets are called closed sets.

3. Generalized rough sets based on neighborhood systems

In this section, we introduce a study of rough sets based on the core of neighborhood systems induced by an arbitrary binary relation. We define four different pairs of dual approximation operators. Also, we compare between our approach and some others approaches.

Definition 3.1. Let U be a non empty set, R be an arbitrary binary relation on U. Then, we can define four types of the core of neighborhood systems induced by R as follows:

(i) The core of right neighborhood($CN_r(x)$): $CN_r(x) = \{y \in U : RN(x) = RN(y)\}$.
(ii) The core of left neighborhood($CN_l(x)$): $CN_l(x) = \{y \in U : LN(x) = LN(y)\}$.
(iii) The core of union neighborhood($CN_u(x)$): $CN_u(x) = CN_r(x) \cup CN_l(x)$.
(iv) The core of intersection neighborhood($CN_i(x)$): $CN_i(x) = CN_r(x) \cap CN_l(x)$.

Definition 3.2. Let U be a non empty set, R be an arbitrary binary relation on U and $CN_j(x)$ be the core of neighborhood systems where $j \in \{r, l, u, i\}$ and $x \in U$. Then (U, R, CN_j) is called an approximation space based on neighborhood induced by the binary relation R (briefly called CN_j-approximation space).

Remark 3.1. Let (U, R, CN_j) be a CN_j-approximation space. If R is an equivalence relation, then the right and left neighborhoods are identical for each element of U. Therefore, $CN_j(x) = [x]_R$ for all $j \in \{r, l, u, i\}$, where $[x]_R$ is the equivalence class of $x \in U$ induced by R. Consequently, our approach is considered a generalization to Pawlak's approximation space.

Lemma 3.1. *Let (U, R, CN_j) be a CN_j-approximation space. Then:*

(i) *$x \in CN_j(x)$ for all $x \in U$ and $j \in \{r, l, u, i\}$.*
(ii) *if $y \in CN_j(x)$. Then $CN_j(x) = CN_j(y)$, for all $x, y \in U$ and $j \in \{r, l, u, i\}$.*

Proof. The proof is obvious from Definition 3.1. □

Definition 3.3. Let (U, R, CN_j) be a CN_j-approximation space and $X \subseteq U$. For each $j \in \{r, l, u, i\}$ and $x \in U$, we define the CN_j-lower approximation and the CN_j-upper approximation of X respectively, as follows:

(i) $\underline{\aleph}_j(X) = \bigcup\{CN_j(x) : CN_j(x) \subseteq X\}$.
(ii) $\overline{\aleph}_j(X) = \bigcup\{CN_j(x) : CN_j(x) \cap X \neq \emptyset\}$.

Definition 3.4. Let (U, R, CN_j) be a CN_j-approximation space, $X \subseteq U$. Then, the subset X is called CN_j-exact set if $\underline{\aleph}_j(X) = \overline{\aleph}_j(X) = X$ for all $j \in \{r, l, u, i\}$. Otherwise, the subset X is called CN_j-rough.