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Original Article

Stress-strength reliability for general bivariate distributions

Alaa H. Abdel-Hamid*

Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

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Abstract An expression for the stress-strength reliability $R = P(X_1 < X_2)$ is obtained when the vector (X_1, X_2) follows a general bivariate distribution. Such distribution includes bivariate compound Weibull, bivariate compound Gompertz, bivariate compound Pareto, among others. In the parametric case, the maximum likelihood estimates of the parameters and reliability function R are obtained. In the non-parametric case, point and interval estimates of R are developed using Govindarajulu's asymptotic distribution-free method when X_1 and X_2 are dependent. An example is given when the population distribution is bivariate compound Weibull. Simulation is performed, based on different sample sizes to study the performance of estimates.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 62G05; 62H10; 62H12; 62N02

1. Introduction

Research on stress-strength model and its generalizations has been collected in [1]. Several papers estimated the stress-strength reliability $R = P(X_1 < X_2)$ when the stress (or supply) X_1 and strength (or demand) X_2 are independent, in the frequentist and Bayes cases. See for example [2-13], among others. Estimation of R when (X_1, X_2) follows a bivariate exponential distribution is discussed in Chapter 3 in [1] and the references therein.

Estimation of R in the non-parametric set up was studied by several authors. See for example [14-19], among others. AL-Hussaini et al. [20] considered parametric estimation of R when X_1 and X_2 are independent and each of which is a finite mixture of lognormal components. Point and interval estimates were obtained and compared in the parametric versus non-parametric cases.

In this paper, R is estimated when the vector (X_1, X_2) follows a general bivariate distribution.

The rest of the paper is organized as follows: A univariate and bivariate distributions are given in Section 2. The model of stress-strength reliability is described in Section 3. Section 4

* Tel.: +201006853842.
E-mail address: hamid_alh@science.bsu.edu.eg
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deals with maximum likelihood and non-parametric estimations of R. Simulation study with illustrations followed by concluding remarks are given in Section 5.

2. Univariate and bivariate distributions

AL-Hussaini and Ateya [21], constructed multivariate distribution by compounding $L(\theta;\ \mathbf{x})$ with $\pi(\theta)$, where

$$L(\theta;\ \mathbf{x}) = \prod_{i=1}^{n} f_{X_i|\Theta}(x_i|\theta), \tag{1}$$

$\mathbf{x} = (x_1, \ldots, x_n)$, $\theta \in \Omega$ is a one-dimensional parameter that belongs to a parameter space Ω,

$$f_{X_i|\Theta}(x_i|\theta) = \delta_i \theta\, z'_{\eta_i}(x_i)\ \exp[-\theta\,\delta_i\, z_{\eta_i}(x_i)],\quad 0 \le a < x_i < b \le \infty, \tag{2}$$

$z_{\eta_i}(x_i)$ is such that $f_{X_i|\Theta}(x_i|\theta)$ is a probability density function (PDF), θ, $\eta_i > 0$, a and b are positive real numbers such that a may assume the value 0 and b the value ∞.

The function $\pi(\theta)$ is given by

$$\pi(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1} e^{-\beta\theta},\quad \theta > 0,\ (\alpha,\ \beta > 0). \tag{3}$$

By compounding $L(\theta;\ \mathbf{x})$ with $\pi(\theta)$, given by (1) and (3), we obtain

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \int_0^{\infty} L(\theta;\ \mathbf{x})\,\pi(\theta)\,d\theta = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\left[\prod_{i=1}^{n}\gamma_i z'_{\eta_i}(x_i)\right]\left[1+\sum_{i=1}^{n}\gamma_i z_{\eta_i}(x_i)\right]^{-\alpha-n}, \tag{4}$$

where

$\gamma_i = \delta_i/\beta > 0$, α, $\eta_i > 0$, $0 \le a < x_i < b \le \infty$, $i = 1, \ldots, n$.

- If, in (4), $n=1$, we obtain

$$f_{X_1}(x_1) = \alpha\,\gamma_i z'_{\eta_1}(x_1)[1+\gamma_1 z_{\eta_1}(x_1)]^{-\alpha-1},\quad x_1 > 0. \tag{5}$$

- For ℓ_1=1, 2, $E(X_1^{\ell_1})$ is given by

$$E\left(X_1^{\ell_1}\right) = \alpha\int_0^1\left[z_{\eta_1}^{-1}\left(\frac{1-w_1}{\gamma_1 w_1}\right)\right]^{\ell_1} w_1^{\alpha-1}\,d w_1, \tag{6}$$

- where $z_{\eta_1}^{-1}(.)$ is the inverse function of $z_{\eta_1}(.)$.
- If, in (4), $n=2$, we obtain

$$f_{X_1,X_2}(x_1, x_2) = \alpha(\alpha+1)\times\left[\prod_{i=1}^{2}\gamma_i z'_{\eta_i}(x_i)\right]\left[1+\sum_{i=1}^{2}\gamma_i z_{\eta_i}(x_i)\right]^{-\alpha-2}. \tag{7}$$

- So that, for $\ell_1 = 1, 2, \ldots,$ and $\ell_2 = 1, 2, \ldots,$ we obtain

$$E\left(X_1^{\ell_1} X_2^{\ell_2}\right) = \alpha(\alpha+1)\times\int_0^1\int_0^1\left[z_{\eta_1}^{-1}\left(\frac{1-w_1}{\gamma_1 w_1}\right)\right]^{\ell_1}\left[z_{\eta_2}^{-1}\left(\frac{1-w_2}{\gamma_2 w_2}\right)\right]^{\ell_2}\times[w_1+w_2-w_1 w_2]^{-\alpha-2} w_1^{\alpha} w_2^{\alpha}\,d w_1\,d w_2. \tag{8}$$

3. stress-strength reliability model

An expression for the stress-strength reliability R is given by the following theorem.

Theorem 3.1. *Suppose that a bivariate PDF of the vector (X_1, X_2) is given by (7). Then*

$$R = P(X_1 < X_2) = 1 - I, \tag{9}$$

where

$$I = \alpha\int_0^1 w^{\alpha-1}\left[1+\gamma_1 w\, z_{\eta_1}\left(z_{\eta_2}^{-1}\left(\frac{1-w}{\gamma_2 w}\right)\right)\right]^{-\alpha-1} d w. \tag{10}$$

Proof
Notice that

$$P(X_1 < X_2) = \int_0^{\infty}\int_0^{x_2} f_{X_1,X_2}(x_1, x_2)\,d x_1\,d x_2 = \alpha(\alpha+1)\int_0^{\infty}\gamma_2 z'_{\eta_2}(x_2)\times[1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha-2} I(x_2)\,d x_2,$$

where

$$I(x_2) = \int_0^{x_2}\gamma_1 z'_{\eta_1}(x_1)[1+A(x_2)\,z_{\eta_1}(x_1)]^{-\alpha-2} d x_1,\quad A(x_2) = \gamma_1[1+\gamma_2 z_{\eta_2}(x_2)]^{-1}. \tag{11}$$

Let $v = [1+A(x_2)\,z_{\eta_1}(x_1)]^{-1}$. Then $z_{\eta_1}(x_1) = \frac{1}{A(x_2)}(\frac{1}{v} - 1)$. Therefore, $\frac{d v}{A(x_2)\,v^2} = -z'_{\eta_1}(x_1)\,d x_1$ and $(0, x_2) \to (1, v_0)$, $v_0 = [1+A(x_2)\,z_{\eta_1}(x_2)]^{-1}$. So that $I(x_2) = \frac{\gamma_1}{A(x_2)}\int_{v_0}^{1} v^{\alpha}\,d v = \frac{\gamma_1}{(\alpha+1)A(x_2)}\{1 - [1+A(x_2)\,z_{\eta_1}(x_2)]^{-\alpha-1}\}$.

Then

$$P(X_1 < X_2) = \alpha\int_0^{\infty}\gamma_2 z'_{\eta_2}(x_2)[1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha-1}\times\left\{1-[1+A(x_2)\,z_{\eta_1}(x_2)]^{-\alpha-1}\right\} d x_2.$$

Notice, from (11), that $\frac{\gamma_1}{A(x_2)} = 1+\gamma_2 z_{\eta_2}(x_2)$. Hence

$$P(X_1 < X_2) = \alpha\left\{\frac{[1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha}}{-\alpha}\Bigg|_0^{\infty}\right\} - I = 1 - I,$$

where

$$I = \alpha\int_0^{\infty}\gamma_2 z'_{\eta_2}(x_2)[1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha-1}\times[1+A(x_2)\,z_{\eta_1}(x_2)]^{-\alpha-1} d x_2.$$

Let

$$w = [1+\gamma_2 z_{\eta_2}(x_2)]^{-1}. \tag{12}$$

Then, from (12)

$$\frac{1}{w} - 1 = \gamma_2 z_{\eta_2}(x_2) \text{ and } \frac{-d w}{w^2} = \gamma_2 z'_{\eta_2}(x_2)\,d x_2.$$