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Original Article

An interior-point penalty active-set trust-region algorithm

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Abstract In this work, an active set strategy is used together with a Coleman–Li strategy and penalty method to transform a general nonlinear programming problem with bound on the variables to unconstrained optimization problem with bound on the variables. A trust-region globalization strategy is used to compute a step. A global convergence theory for the proposed algorithm is presented under credible assumptions.

Prefatory numerical experiment on the algorithm is presented. The rendering of the algorithm is reported on some classical problem.

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1. Introduction

In this paper, we consider the following constrained optimization problem

$$\begin{array}{lllll} \textit{minimize} & f(x) & & & \\ \textit{subject to} & a_i(x) & = & 0 & i \in E, \\ & a_i(x) & \leq & 0 & i \in I, \\ & \alpha \leq x \leq \beta, & & & \end{array} \tag{1.1}$$

where $f: \Re^n \to \Re$, $a_i: \Re^n \to \Re^m$, $E \bigcup I = \{1, ..., m\}$ and $E \bigcap I = \emptyset$, $\alpha \in \{\Re \bigcup \{-\infty\}\}^n$, $\beta \in \{\Re \bigcup \{\infty\}\}^n$, $m < n$, and $\alpha < \beta$. The functions f and a_i, $i = \{1, ..., m\}$ are presumed to be at least twice continuously differentiable. We denote the feasible set $\boldsymbol{F} = \{x : \alpha \leq x \leq \beta\}$ and the strict interior feasible set $int(\boldsymbol{F}) = \{x : \alpha < x < \beta\}$.

In this paper, we use an active-set strategy in [1] to convert the above problem to an equality constrained optimization problem with bounded variables. The head feature of the suggested active set is that it is identified and updated naturally by the step. See [2–4].

A penalty method is used in this paper to transform the equality constrained optimization problem which was obtained from the above step to unconstrained optimization problem with bound on variables. Some penalty functions have been suggested and many contributions addressing the convergence of these methods have been made, see [5,6].

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A Coleman–Li strategy in [7] is used to form a sequential quadratic programming subproblem of unconstrained optimization problem. For more details, see [7–9].

In this paper, we use a trust-region strategy to evaluate a step. A trust-region strategy is globalization method which means modifying the local method in such a way that it is ensured to converge even if the starting point is far away from the solution. Most trust-region algorithms for solving a constrained optimization problem try to merge the trust-region idea with the sequential quadratic programming method. See [2–4,10]. Under credible assumptions, a convergence theory for our algorithm is introduced.

The rest of this section introduces some notations that are used throughout the rest of the paper. The paper is arranged as follows. In Section 2, a detailed characterization of the main steps to form a sequential quadratic programming subproblem is introduced. In Section 3, a detailed characterization of an interior-point trust-region algorithm is given. Sections 4–9 are devoted to the global convergence theory of the proposed algorithm under important assumptions. Section 10 contains a Matlab implementation of the interior-point trust-region algorithm and our numerical results. Finally, Section 11 contains concluding remarks.

In this paper, we use the symbol $f_k = f(x_k)$, $\nabla f_k = \nabla f(x_k)$, $\nabla^2 f_k = \nabla^2 f(x_k)$, $A_k = A(x_k)$, $\nabla A_k = \nabla A(x_k)$, $Z_k = Z(x_k)$, $W_k = W(x_k)$ and so on to denote the function value at a particular point. We denote to the Hessian of the objective function f_k or an approximation to it by H_k . Finally, all norms are l_2-norms.

2. A sequential quadratic subproblem

Motivated by the active-set strategy in [1], we define a 0–1 diagonal matrix $W(x) \in \Re^{m\times m}$ whose diagonal entries are

$$w_i(x) = \begin{cases} 1, & \text{if } i \in E, \\ 1, & \text{if } i \in I \text{ and } a_i(x) \geq 0, \\ 0, & \text{if } i \in I \text{ and } a_i(x) < 0. \end{cases} \tag{2.1}$$

Using the above matrix, problem (1.1) is converted to the following

$$\begin{aligned} \text{minimize} \quad & f(x), \\ \text{subject to} \quad & A(x)^T W(x) A(x) = 0, \\ & \alpha \leq x \leq \beta, \end{aligned}$$

where $A(x) = (a_1(x), ..., a_m(x))^T$ is a continuously differentiable function.

Using a penalty method, the above problem is transformed to the following unconstrained optimization problem with bounds on the variable

$$\begin{aligned} \text{minimize} \quad & f(x) + \frac{r}{2}\|W(x)A(x)\|^2, \\ \text{subject to} \quad & \alpha \leq x \leq \beta, \end{aligned} \tag{2.2}$$

where $r > 0$ is a penalty parameter. Let

$$\phi(x; r) = f(x) + \frac{r}{2}\|W(x)A(x)\|^2. \tag{2.3}$$

The Lagrangian function associated with bounded problem (2.2) is given by

$$L(x, \lambda, \mu; r) = \phi(x; r) - \lambda^T(x - \alpha) - \mu^T(\beta - x), \tag{2.4}$$

where λ and μ are Lagrange multiplier vectors associated with the inequality constraints $x - \alpha \geq 0$ and $\beta - x \geq 0$ respectively.

The first-order necessary conditions for a point x_* to be a solution of problem (1.1) are the existence of multipliers $\lambda_* \in \Re^n_+$, and $\mu_* \in \Re^n_+$, such that (x_*, λ_*, μ_*) satisfies

$$\nabla\phi(x_*; r_*) - \lambda_* + \mu_* = 0, \tag{2.5}$$

$$\alpha \leq x_* \leq \beta, \tag{2.6}$$

and for all j corresponding to $x^{(j)}$ with finite bound, we have

$$\lambda_*^{(j)}(x_*^{(j)} - \alpha^{(j)}) = 0, \tag{2.7}$$

$$\mu_*^{(j)}(\beta^{(j)} - x_*^{(j)}) = 0, \tag{2.8}$$

where $\nabla\phi(x_*; r_*) = \nabla f(x_*) + r_*\nabla A(x_*)W(x_*)A(x_*)$.

Let $Z(x)$ be the diagonal scaling matrix whose diagonal elements are given by

$$z^{(j)}(x) = \begin{cases} \sqrt{(x^{(j)} - \alpha^{(j)})}, & \text{if } (\nabla\phi(x; r))^{(j)} \geq 0 \text{ and } \alpha^{(j)} > -\infty, \\ \sqrt{(\beta^{(j)} - x^{(j)})}, & \text{if } (\nabla\phi(x; r))^{(j)} < 0 \text{ and } \beta^{(j)} < +\infty, \\ 1, & \text{otherwise.} \end{cases} \tag{2.9}$$

For more details see [7,8].

Using the diagonal scaling matrix $Z(x)$, the first order necessary conditions for the point x_* to solve problem (1.1) are that $x_* \in \boldsymbol{F}$ and solves the following nonlinear system

$$Z^2(x)\nabla\phi(x; r) = 0. \tag{2.10}$$

Any point $x_* \in \boldsymbol{F}$ that satisfies the condition (2.10) is called a Karush–Kuhn–Tucker point or KKT point. For more details see [5].

A system (2.10) is continuous but not differentiable at some point $x \in \boldsymbol{F}$. The non-differentiability happens when $z^{(j)} = 0$ and these points are averted by restricting $x \in int\boldsymbol{F}$. Also the non-differentiability happens when a variable $x^{(j)}$ has a finite lower bound and an infinite upper bound and $(\nabla\phi(x; r))^{(j)} = 0$. But these points are not significant, so we define a vector $\psi(x)$ whose components are $\psi^{(j)}(x) = \frac{\partial((z^{(j)})^2)}{\partial x^{(j)}}$, $j = 1, ..., n$ such that $\psi^{(j)}$ to be zero whenever $(\nabla\phi(x; r))^{(j)} = 0$. Hence, we can write

$$\psi^{(j)}(x) = \begin{cases} 1, & \text{if } (\nabla\phi(x; r))^{(j)} \geq 0 \text{ and } \alpha^{(j)} > -\infty, \\ -1, & \text{if } (\nabla\phi(x; r))^{(j)} < 0 \text{ and } \beta^{(j)} < +\infty, \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

Assuming $x \in int(\boldsymbol{F})$ and applied Newton's method on the system (2.10), then we have

$$\begin{aligned} &[Z^2(x)\nabla^2\phi(x; r) + diag(\nabla\phi(x; r))diag(\psi(x))]\Delta x \\ &\quad = -Z^2(x)\nabla\phi(x; r), \end{aligned} \tag{2.12}$$

where

$$\nabla^2\phi(x; r) = H + r\nabla A(x)W(x)\nabla A(x)^T, \tag{2.13}$$

and H is the Hessian of the objective function $f(x)$ or an approximation to it. Multiplying both sides of Eq. (2.12) by $Z^{-1}(x)$ and scale the step using $\Delta x = Z(x)s$, then we have

$$\begin{aligned} &[Z(x)\nabla^2\phi(x; r)Z(x) + diag(\nabla\phi(x; r))diag(\psi(x))]s \\ &\quad = -Z(x)\nabla\phi(x; r), \end{aligned} \tag{2.14}$$

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