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New classes of analytic functions determined by a modified differential-difference operator in a complex domain

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Abstract

In the recent study, we utilize a differential-difference operator in a complex domain to define some new classes of analytic functions. A modification of the differential-difference operator in the open unit disk is formulated in order to impose some geometric properties. The modified operator is studied in the Hardy space. Additionally, we define a new subspace of the Hardy space comprising the normalized analytic functions with the set of all bounded functions. We shall validate that the modified operator is closed in the subspace of normalized functions with the bounded first derivative. Our proofs are based on the Jack Lemma.

Keywords: Differential operator; Unit disk; Analytic function; Subordination; Superordination

1. Introduction

The concept of derivatives of functions plays a significant role in the theory of mathematical analysis and its branches, especially the geometric function theory. This brings us to find out some properties of a function and its behavior. The derivatives of a function can be formulated in various methods, such as classes, operators, equations (differential) and inequalities. Our study is in the differential operator theory. This theory deals with operators that defined by derivatives of the functions. Simply, the differential operator is an operator with derivatives. There are many applications in this direction, it can be seen in various parts of science, engineering and medicine. Moreover, applications can be found in social, music and art studies. There are two types of the differential operators depending on the definition of the derivative. These types are analytic and difference form. Our investigation is about the mixed differential-difference operators.

In 1989 Dunkl [1] introduced a differential-difference operator, which is connected to the theory of sampling signals (multi-body systems). The dynamical features of this operator are investigated in many literatures such as [2]. Dunkl operator characterizes a significant generalization of partial derivatives and achieves the commutative law in R^n. In geometry, it achieves the reflexive relation, which is mapping the space into itself as a set of fixed points (see Refs. [3–5]).

In this study, we utilize a differential-difference operator in a complex domain to define some new classes of analytic functions. A modification of the

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differential-difference operator in the open unit disk is formulated in order to impose some geometric properties. The modified operator is studied in the Hardy space. Additionally, we define a new subspace of the Hardy space comprising the normalized analytic functions with the set of all bounded functions. We shall validate that the modified operator is closed in the subspace of normalized functions with the bounded first derivative. Our proofs are based on the Jack Lemma.

2. Processing

Let $U := \{z : |z| < 1\}$ be the open unit disk of the complex plane and $H(U)$ be the space of holomorphic functions on the open unit disk. A holomorphic function

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U$$

on the open unit disk belongs to the Hardy space $H^2(U)$, if its sequence of power series coefficients is square-summable:

$$H^2(U) = \left\{ \phi \in H(U) : \sum_{n=0}^{\infty} |\varphi_n|^2 < \infty \right\}.$$

Consequently, it can be defined a norm on $H^2(U)$ as follows [5]:

$$\|\phi\|^2_{H^2(U)} = \sum_{n=0}^{\infty} |\varphi_n|^2.$$

Since $L^2(U)$ is Banach space, then $H^2(U)$ is also a Banach space on U. In the sequel, we deal with a subset of analytic function, which are normalized as follows: $\phi(0) = 0$ and $\phi'(0) = 1$. Therefore, ϕ is defined as follows:

$$\phi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad z \in U.$$

We denote this class by $A(U)$. For $\phi,\psi \in A(U)$, the convolution product, (*) is defined by

$$(\phi * \psi)(z) = \left(z + \sum_{n=2}^{\infty} \varphi_n z^n \right) * \left(z + \sum_{n=2}^{\infty} \vartheta_n z^n \right) = z + \sum_{n=2}^{\infty} \varphi_n \vartheta_n z^n.$$

It is clear that $A(U) \subset H(U)$ achieving the above norm. The space H^∞ is known as the vector space of bounded holomorphic functions on U, achieving the norm

$$||\phi||_{H^\infty} = \sup_{|z|<1} |\phi(z)|, \quad \phi \in H(U).$$

Obviously,

$$H^2(U) \subset H^\infty, \quad \phi \in H(U).$$

We wish to establish a modified differential-difference operator, based on the Dunkle operator in the open unit disk. First, the Dunkle operator is defined as follows on the space of holomorphic functions ($\Lambda_\alpha : H(U) \to H(U)$):

$$(\Lambda_\alpha \phi)(z) = \phi'(z) + \frac{2\alpha+1}{z} \left(\frac{\phi(z) - \phi(-z)}{2} \right) \quad \left(z \in U \setminus \{0\}, \phi \in H(U), ' = \frac{d}{dz}, \alpha > \frac{-1}{2} \right). \tag{1}$$

Note that for $\phi \in A(U)$, we have

$$(\Lambda_\alpha \phi)(z) = 2(\alpha+1) + \sum_{n=2}^{\infty} A_{n,\alpha} z^{n-1}, \quad z \in U.$$

We proceed to define the Dunkle operator in the class $A(U)$. For $\phi \in A(U)$, we construct the modified Dunkle operator $\mathrm{T}_\alpha : A(U) \to A(U)$

$$(\mathrm{T}_\alpha \phi)(z) := \frac{z(\Lambda_\alpha \phi)(z)}{2(\alpha+1)} = \frac{z\left[\phi'(z) + \frac{2\alpha+1}{z}\left(\frac{\phi(z)-\phi(-z)}{2}\right)\right]}{2(\alpha+1)} = z + \sum_{n=2}^{\infty} \lambda_n z^n, \tag{2}$$

where

$$\lambda_2 = \frac{\varphi_2}{\alpha+1}, \quad \lambda_3 = \frac{(\alpha+2)\varphi_3}{\alpha+1}, \ldots.$$

In general, we have

$$\lambda_n = \begin{cases} \dfrac{n}{2(\alpha+1)} \varphi_n & \text{if } n \text{ is even} \\ \dfrac{2\alpha+n+1}{2(\alpha+1)} \varphi_n & \text{if } n \text{ is odd} \end{cases}.$$

Obviously, $\mathrm{T}_\alpha \in A(U)$. Let $S^2(U)$ be the space defined by

$$S^2(U) := \left\{ \phi \in H(U) : \phi' \in H^2(U) \right\}$$

end with the norm

$$||\phi||^2_{S^2(U)} := ||\phi||^2_{H^2(U)} + \|\phi'\|^2_{H^2(U)}.$$

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