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Approximate solutions to nonlinear oscillations via an improved He's variational approach

M.K. Yazdi

School of Mechanical Engineering, Iran University of Science and Technology, Narmak, Tehran, 16846, Iran

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Abstract

This study investigates accuracy of the He's variational approach with a rational trial function to obtain the frequency of conservative nonlinear oscillations. This method utilizes a semi-inverse method to establish variational principles from the governing equations. Four illustrative examples are considered, the restoring force of the studied cases has non-integer and cubic power terms as well as discontinuities. The last case considers the mathematical model of nonlinear oscillations of an elevator cable. The results demonstrate the usefulness and merit of the algorithm.

Keywords: Nonlinear oscillation; He's variational approach; Rational trial function; Analytical approximate solution

1. Introduction

Nonlinear oscillations arise in various areas of engineering and applied sciences [1]. The accurate prediction of it has been a notable topic. Therefore, numerous researchers have been exerted perturbative and non-perturbative procedures to find a precise amplitude−frequency relationship. El-Borgi et al. [2] considered the free and forced vibration response of simply-supported functionally graded (FG) nanobeams resting on a non-linear elastic foundation using the direct and discretized method of multiple scales (MMS). Elmas and Boyaci [3] introduced a perturbation algorithm using a new variable transformation. Daeichin et al. [4] employed a rational initial guess and combined it with energy balance and collocation methods for solving nonlinear oscillators with cubic term. Kanani et al. [5] investigated the effect of nonlinear elastic foundation on the frequency and time responses by the variational iteration method (VIM). Belendez et al. [6] analyzed the Duffing oscillator using a modified rational harmonic balance method. Younesian et al. [7] employed the frequency−amplitude formulation and the energy balance method to solve the generalized Duffing equation. Cveticanin et al. [8] focused on solving a generalized second-order strongly nonlinear differential equation which describes the motion of a conservative oscillator with restoring force of series type with integer and non-integer displacement functions. Yazdi and Tehrani [9] improved accuracy of the energy balance via Jacobi collocation method to conservative nonlinear oscillators. Khan et al. [10] carried out a frequency analysis

E-mail address: ckalami@gmail.com.
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on a generalized conservative nonlinear equation utilizing two kinds of the frequency—amplitude formulation. Sedighi and Shirazi [11] used the parameter expansion method to predict the nonlinear vibrational behavior of micro-beams actuated by an electric field.

He [12] introduced the variational approach to determine the frequency—amplitude relation of some nonlinear oscillators using the semi-inverse [13] and the Ritz's methods. Thenceforth, researchers successfully applied the approach to several nonlinear problems [14—17]. Among the applications of this approach, one may refer to solutions of the generalized Zakharov [18,19], the Thomas-Fermi [20], the Benjamin—Bona—Mahony and the Kawahara equations [21]. Moreover, it is effective for studying on nonlinear vibration of an eccentrically reinforced cylindrical shell [22], investigation of the nonlinear frequency of protein microtubules embedded in the cytoplasm [23], analysis of piezoelectric nanobeams [24], obtaining natural frequency of nonlocal functionally graded (FG) beams resting on nonlinear elastic foundation [25] and closed-form solutions for the nonlinear vibration frequency of FG microplates based on the modified couple stress theory [26]. In additional, He proposed different asymptotic techniques such as the homotopy perturbation method [27,28], the max—min approach [29,30], the amplitude—frequency formulation [7,31] and the variational iteration method [5,32] to weakly & strongly nonlinear equations. It should be noted an elementary review regarding to these approaches can be found through Refs. [33—35].

Mickens [36] provided a general framework for determining higher order corrections to the method of harmonic balance using a rational trial function. The aim of the current study is to examine the applicability and suitability of the rational trial function with the variational approach. The results reveal the combination causes a good accuracy to the studied nonlinear oscillators.

The rest of the manuscript is arranged as follows. Summary of the rational variational approach (RVA) is given in Section 2. The studied cases are scrutinized in Section 3. Section 4 ends the study with a brief corollary.

2. Solution procedure

This section overviews the outline of the rational variational approach (RVA). Consider a general nonlinear oscillator as follows:

$$u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{1}$$

By assuming a rational trial function as follows:

$$u(t) = A(1+B)\cos(\omega t)(1+B\cos(2\omega t))^{-1}, \tag{2}$$

the variational form of Eq. (1) can be established as follows:

$$J = \int_0^t \overbrace{\left(-\frac{1}{2}u'^2 + F(u)\right)}^{G(A,B,\omega,t)} dt, \quad \partial F/\partial u = f. \tag{3}$$

where $G(A,B,\omega,t)$ is a gained function when Eq. (2) is substituted into Eq. (3).

As $|B| \ll 1$, it is possible to replace G with its Taylor series expansion as follows:

$$G(A,B,\omega,t) = \sum_{z=0}^{\infty} g_z(A,\omega,t)B^z, \tag{4}$$

where

$$g_z(A,\omega,t) = \frac{1}{z!}\left(\frac{\partial^z G(A,B,\omega,t)}{\partial B^z}\right)_{\mathbf{B}=0}. \tag{5}$$

This study considers the following approximation:

$$\begin{aligned}\widehat{G}(A,B,\omega,t) &= \sum_{z=0}^{4} g_z(A,\omega,t)B^z \\ &= g_0(A,\omega,t) + g_1(A,\omega,t)B + \cdots \\ &\quad + g_4(A,\omega,t)B^4.\end{aligned} \tag{6}$$

Therefore, Eq. (3) reduces to:

$$\widehat{J} = \int_0^{T/4} \widehat{G} dt, \quad T = 2\pi\omega^{-1}. \tag{7}$$

By setting $\frac{\partial \widehat{J}}{\partial A} = 0$ & $\frac{\partial \widehat{J}}{\partial B} = 0$, one may attain a pair of frequencies (i.e., $\omega^{(1)}$ & $\omega^{(2)}$). By equating the frequencies for a value of initial amplitude, the parameter of B that generally depends on A is acquired. To elucidate the capability of the approach, the technique is exercised to some cases in the next section. The outcomes reveal the present method has an excellent accuracy.

3. Examples

The rational variational approach (RVA) is applied to four cases in this section. The restoring force of the first case has a fractional power term. An antisymmetric form is considered in the second case and the third case corresponds to the cubic Duffing with discontinuity. The last case displays the mathematical model of nonlinear oscillations of an elevator cable in a