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Solution method for a non-homogeneous fuzzy linear system of differential equations

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ABSTRACT

In this paper, we propose a new solution method to non-homogeneous fuzzy linear system of differential equations. The coefficients of the considered system are crisp while forcing functions and initial values are fuzzy. We assume each forcing function be in a special form, which we call as triangular fuzzy function and which represents a fuzzy bunch (set) of real functions. We construct a solution as a fuzzy set of real vector-functions, not as a vector of fuzzy-valued functions, as usual.

We interpret the given fuzzy initial value problem (fuzzy IVP) as a set of crisp (classical) IVPs. Such a crisp IVP is obtained if we take a forcing function from each of fuzzy bunches and an initial value from each of fuzzy intervals. The solution of the crisp IVP is a vector-function. We define it to be an element of the fuzzy solution set and assign a membership degree which is the lowest value among membership degrees of taken forcing functions and initial values in the corresponding fuzzy sets.

We explain our approach and solution method with the help of several illustrative examples. We show the advantage of our method over the differential inclusions method and its applicability to real-world problems.

1. Introduction

Dynamic processes, which do not contain the randomness but some parameters of which are uncertain, can be modelled adequately by using fuzzy sets. Some of such models naturally lead to fuzzy differential equations and, in particular, to systems of fuzzy differential equations [3,14,18,19]. There are many important studies on fuzzy differential equations [11–17,31,35–40,45,46,52]. A variety of different methods have been proposed in these studies. These methods differ depending on the concepts of fuzzy function and fuzzy derivative used.

One of the first works on studying the fuzzy differential equation systems belongs to Oberguggenberger and Pittschmann [47]. They apply the Zadeh extension principle to the system of differential equations with fuzzy parameters and introduce the notions of fuzzy solutions and component-wise fuzzy solutions. Buckley et al. [14] propose two close methods for the solution of linear systems of first-order differential equations with fuzzy initial conditions. In the first method, the authors fuzzify the crisp solution and then check to see if its alpha-cuts satisfy the differential equations. In the second method, they solve the fuzzified level-wise system and then check if the solution always (i.e. for all t) defines a valid fuzzy number or not. Unfortunately, a solution of type 1 or type 2, defined such a way, exists only for specific systems. Xu et al. [53] also investigate linear first-order fuzzy differential equations systems with fuzzy initial values. They use the complex number representation for the alpha-level sets, which was proposed firstly by Pearson [48], and prove existence theorems of solutions. The authors also describe phase portraits of the two-dimensional fuzzy dynamical systems. Nieto et al. [45] study the existence and uniqueness of solution for fuzzy differential systems under Hukuhara differentiability. Xu et al. [54] study the properties of first-order linear dynamical systems with fuzzy matrices. They construct the fuzzy solution from the solutions of the classical differential equations, obtained by using the alpha-level representation for the fuzzy system. Gasilov et al. [26] develop a geometric method to solve fuzzy linear systems of differential equations. Fard and Ghal-Eh [23] propose an iterative method to obtain approximate solution for the linear systems of first-order differential equations with fuzzy constant coefficients. Ghazanfari et al. [32] investigate linear first-order fuzzy matrix differential equation systems using the complex number representation for the alpha-level sets and prove some properties of the matrix differential systems. Mosleh and Otadi [43] propose a method for finding a minimal solution of a system of

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fuzzy linear differential equations in the form $A\dot{x}(t) = B\dot{x}(t) + Cx(t)$. Hashemi et al. [34] apply the homotopy analysis method to derive an approximate analytical solution for the system of fuzzy differential equations. Mosleh [42] presents a neural network to solve system of fuzzy differential equations with fuzzy initial values.

The vast majority of studies [1–3,6,9,10,15,39,50,51], which have been carried out in recent years, use the generalized derivative, proposed by Bede and Gal [8]. But this derivative concept leads to some difficulties:

(1) The generalized derivative is mainly a combination of the 1-derivative (or, the Hukuhara derivative) and the 2-derivative (or, second type Hukuhara derivative). Under the 1-derivative we have a solution, the uncertainty (fuzziness) of which increases with the time. Contrary to this, under the 2-derivative we have a solution with decreasing uncertainty. To describe a solution, the uncertainty of which alternates, we have to alternate 1- and 2-derivatives. Consequently, we have a priori to divide the time domain into subintervals. Into how many subintervals should one divide the time domain? How to select the length of each subinterval? Which derivative should be used in each subinterval? How to address these questions remains an open problem.
(2) When the generalized derivative is used, for an n-dimensional fuzzy system it is necessary to examine a set of classical systems in number of 2^n. This circumstance limits the number of researches, which use the generalized derivative.
(3) In general, the solution of a fuzzy differential equation under the generalized differentiability is not unique. What to do, if we have 2 or more solutions? This question has not been answered yet.

Studying fuzzy differential equation systems is also very important to solve the fuzzy optimal control problems. There are only a few studies on this issue due to the reasons, described above [4,5,24,41,44,49,55].

The motivation of our studies is to develop an approach to fuzzy differential equations, which overcomes the above-mentioned difficulties with fuzzy derivative. The main idea of the approach is that we interpret a fuzzy function as a fuzzy set (bunch) of real functions. In [25], the approach was applied to solve a non-homogeneous linear system of interval differential equations. As it is known, fuzzy sets are generalization of crisp (classical) sets. In particular, fuzzy numbers are generalization of real intervals. Consequently, if there is a solution method for a fuzzy problem, it can be adapted to the associated interval problem. But, in general, the opposite of this statement is not valid, i.e., a method for an interval problem cannot be applied to the corresponding fuzzy problem directly. More general concepts and specific derivations can be required. In this paper, based on results [25], we develop a solution method for non-homogeneous fuzzy linear system of differential equations. In this, we present the solution as a fuzzy bunch of vector-functions. The proposed approach is motivated by its usefulness in avoiding the difficulties with fuzzy derivative. Namely, (1) since it does not use a fuzzy derivative, there is no need to divide the time interval into subintervals, and to perform different operations on different subintervals; (2) the solution exists and is unique; and (3) the complexity is $O(n)$.

The paper is organized as follows. After the Introduction, in Section 2, we emphasize the fact that a fuzzy function can be interpreted by two different ways. We show the difference between them. In Section 3, we consider triangular fuzzy functions (a new concept of fuzzy bunch of real functions). In Section 4, we describe a non-homogeneous fuzzy linear system of differential equations, which we investigate. We give a definition of the solution and develop a solution method. In Section 5, we demonstrate the method on a test example. In Section 6, we compare our solution method with the differential inclusions method. In Section 7, we discuss some application aspects of the subject. In Section 8, we give concluding remarks.

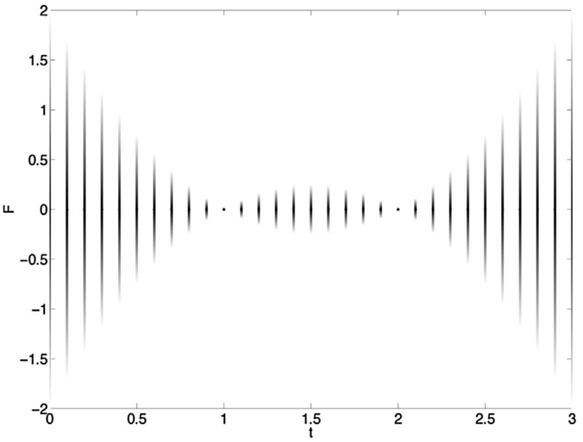

Fig. 1. The values of the triangular-fuzzy-number-valued function $\tilde{F}(t) = (-|t^2 - 3t + 2|,\quad 0,\quad |t^2 - 3t + 2|)$ at different points taken with step $\Delta t = 0.1$.

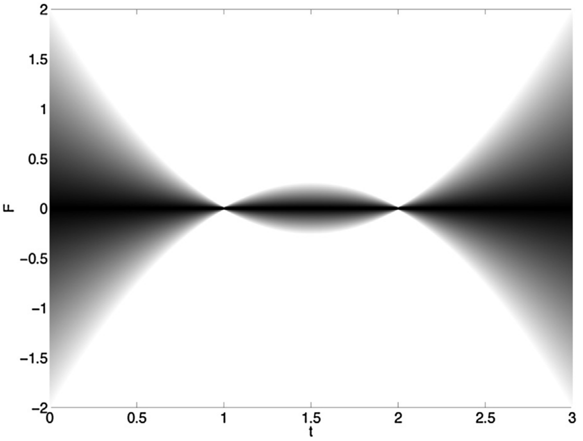

Fig. 2. Continuous representation for fuzzy functions from Examples 1 and 2.

2. Two types of fuzzy functions

In this paper, we interpret a fuzzy function as a fuzzy bunch of real functions [26–29], not as a fuzzy-valued function as usual. What is the difference between of these two types of fuzzy functions we explain using examples.

Example 1. Consider the triangular-fuzzy-number-valued function $\tilde{F} : [0,\quad 3] \to T_R$ (here T_R denotes the set of triangular fuzzy numbers) defined as

$$\tilde{F}(t) = (-|t^2 - 3t + 2|,\quad 0,\quad |t^2 - 3t + 2|)$$

At $t = 0, 0.1, \ldots, 0.9, 1, 1.1, \ldots, 3$ (with step $\Delta t = 0.1$) the values of $\tilde{F}$ are $(-2, 0, 2), (-1.71, 0, 1.71), \ldots, (-0.11, 0, 0.11), (0, 0, 0), (-0.09, 0, 0.09), \ldots, (-2, 0, 2)$, respectively. We depict these values in Fig. 1. In this we use a grayscale image, where black color represents the degree of membership of 1 and white represents 0. If we change t continuously, we have the graph of the given fuzzy function $\tilde{F}$ (see Fig. 2).