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Numerical treatment for boundary value problems of Pantograph functional differential equation using computational intelligence algorithms

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ABSTRACT

In this study, stochastic computational techniques are developed for the solution of boundary value problems (BVPs) of second order Pantograph functional differential equation (PFDE) using artificial neural networks(ANNs), simulated annealing (SA), pattern search (PS), genetic algorithms (GAs), active-set algorithm (ASA) and their hybrid combinations. The strength of ANNs is exploited to construct a model for PFDE by defining as unsupervised error to approximate the solution. The accuracy of the model is subjected to find the appropriate design parameters of the networks. These optimal weights of the networks are trained using SA, PS and GAs, used as a tool for viable global search, hybridized with ASA for rapid local convergence. The designed schemes are evaluated by solving a numbers of BVPs for the PFDE and comparing with standard results. The reliability and effectiveness of the proposed solvers are investigated through Monte-Carlo simulations and their statistical analysis.

Introduction

The functional differential (FD) equations illustrate the properties of dynamical processes for which the motion is dependent on their prior history and anticipated future of the states of these processes. The importance and significance for such processes is well established branch of non-linear analysis. Stability for large scale systems described by FD equations include the model of large class of electrical networks developed for linearly lumped parameter elements and non-linear memory-less elements with *RC* lines [1,2], and the model for population dynamics [3,4]. There are a number of classical, as well as, modern applications addressed by FD equations including stability and buffrication of genetic regulatory networks models [5], identifiability analysis for linear time-delay systems [6], stability of non-linear quasi-monotone dynamical systems [7], direct adaptive control model for parameter information content of measurable signals [8], dynamic model of global exponential stability for Cellular Neural networks [9], models for quantization, time delays, stability of non-linear control systems [10], Non-linear SISO system model to control the tracking error of funnel with saturation [11], and modeling for synchronization of multi-perturbation delay chaotic systems using dual-stage impulsive control [12]. Further applications can be seen elsewhere [13,14].

The well-known Pantograph equations belong to the class of FD equations with proportional delay. The origin of the name Pantograph belongs to Ockendon and Tayler [15] on their premier work for collection of current by the Pantograph head of an electric locomotive. Such type of systems arise in variety of applications of adaptive control, number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structure, quantum mechanics, cell growth, engineering, economics and etc. [15–19]. The research community has been attracted to investigate the solutions of such kind systems since its origin till to date.

The existing numerical solvers for two-point boundary value problem associated with such second order systems are based on finite differences approach [20], Chebyshev polynomials method [21], variational techniques [22], Runge–Kutta–Nystrom methods [23], Homotopy perturbation method [24], spline functions [25], Richardson extrapolation approach [26], Adomian decomposition method [27,28], shooting techniques [29], and piecewise polynomial collocation method [30]. All these techniques have their own strengths and weaknesses over other solvers in terms of accuracy, convergence, reliability and robustness. As far as our literature

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survey is concerned only these type solvers are used for such system while stochastic solvers have not yet been applied in this area.

Applied soft computing techniques based on neural networks, evolutionary computing and swarm intelligence methods have been used extensively for solving linear and non-linear systems associated with ordinary and partial differential equations [31–34]. For example, these methodologies provides reliable solution to Van der Pol nonlinear oscillators [35], Troesch's problem arises in plasma physics [36], Nonlinear singular systems based on Emden Fowler equations [37], first Painlevé transcendent [38], the non-linear Schrodinger equations [39], fluid flow problems [40,41] and Nonlinear Bratu's problems arising in fuel ignition model of combustion theory [42], etc. Interested reader is referred to survey article [43], which summarized the research of numerous scholars mostly within the last decade to get a better knowledge about the present research scenario. Recently, these techniques are also extended to solve the systems based on linear and non-linear ordinary fractional differential equations as well [44,45]. For instant, practical applications of well-known fractional order systems linked with Riccati equation, as well as, Bagley–Torvik equations have also been solved effectively [46,47]. These are motivation factors for the authors to investigate and analyze the heuristic techniques to solve FD equations, particularly the boundary values problems of Pantograph type.

In this paper, numerical solution for the two point boundary value problems (BVP) of Pantograph FD equations is presented using stochastic techniques. The generic form of second order FD equation can be given as below:

$$\begin{cases} \dfrac{d^2y(t)}{dt^2} = f(t, y(t), y(x(t))), \quad t \in (0, T) \\ y(0) = b, \quad y(T) = c \end{cases} \tag{1}$$

where $T > 0$, $(b, c) \in \varphi$ and x: $[0, T] \to \varphi$ is such that $0 \le x(t) \le T$, for $t \in [0,T]$. The case $x(t) = \lambda t$ with λ the proportional factor, corresponds to second order Pantograph equation. To approximate the equation, feed-forward artificial neural networks (ANNs) are appropriately combined to define an objective function. The weights of these networks are optimized to minimize the objective function value with help of SA, PS, GAs, ASA, SA hybridized with ASA (SA-ASA), PS combined with ASA (PS-ASA), GA hybridized with ASA (GA-ASA) techniques. The accuracy, reliability, effectiveness of these schemes is investigated through Monte-Carlo simulations and their statistical analysis. Comprehensive studies of stochastic solvers are made on the basis of time of execution, convergence and level of accuracy achieved. The comparison of the proposed solutions is made with exact solutions, as well as, with recently developed numerical technique based on successive interpolation [48].

Organization of the paper is as follows. In Mathematical Modelling Section, neural networks mathematical model for Pantograph FD equation is presented. In Learning Methodologies Section, introductory material is provided for solvers along with parameter setting used for training of weights of networks. In Numerical Experimentations Section, results are presented for numerical experimentation carried out with proposed schemes. In Comparative Analysis of Results Section, the discussion on comparative analyses and computational complexity of the proposed solvers is provided. Last section concludes our findings and also provides the future recommendations.

Mathematical modeling

In this section, mathematical modeling of Pantograph equation is presented. Linear combinations of feed-forward ANNs are used. Log-sigmoid is used as activation function in neural network architecture.

In case of ordinary differential equations the following continuous mapping is applied to approximate solution $y(t)$, and its derivatives as [49,50]

$$\begin{cases} \hat{y}(t) = \displaystyle\sum_{i=1}^{k} \alpha_i g(w_i t + \beta_i), \\ \dfrac{d^n}{dt^n}\hat{y}(t) = \displaystyle\sum_{i=1}^{k} \alpha_i \dfrac{d^n}{dt^n} g(w_i t + \beta_i), \end{cases} \tag{2}$$

where α, β and w represent the unknown constants, called neural network weights, k is number of neurons and g is log-sigmoid activation function given as $g(t) = 1/(1 - e^{-t})$.

Mathematical model for FD equations can be formulated as an extension of the networks given in Eq. (2) by replacing independent variable t with a function $x(t)$. Therefore, the approximate neural networks models for solution $y(t)$ and its derivatives of the equation can be given in modified form as:

$$\begin{cases} \hat{y}(x(t)) = \displaystyle\sum_{i=1}^{k} \alpha_i g(w_i x(t) + \beta_i), \\ \dfrac{d^n}{dt^n}\hat{y}(x(t)) = \displaystyle\sum_{i=1}^{k} \alpha_i \dfrac{d^n}{dt^n} g(w_i x(t) + \beta_i), \end{cases} \tag{3}$$

Linear combination of the networks given in equations (3) can model the FD equations provided the unknown weights are properly optimized and it is named as FD equations neural (FDEN) network. The network given in equations (3) has a varied response by taking different relation of $x(t)$. For example, by taking $x(t) = t$, the networks for FD Eq. (3) is similar to networks of ordinary differential equations given in (2). By taking $x(t) = \lambda t$ the networks provided in (3) are used to model Pantograph differential equations. Similarly, by using $x(t) = t - \tau$, this network is used to model the τ time delay differential equations. The FDEN network architecture for one of the form of Eq. (1) using $f = t + y(t) + y(x(t))$ is shown graphically in Fig. 1.

The objective function ε is given as:

$$\varepsilon = \varepsilon_1 + \varepsilon_2, \tag{4}$$

where M is the total number of iterations. The value ε_1 is associated with FD equations and it is given below in case of (1) using the networks provided in Eq. (3) as:

$$\varepsilon_1 = \frac{1}{N+1}\sum_{m=0}^{N}\left(\frac{d^2\hat{y}_m}{dt^2} - f(t_m, \hat{y}_m, \hat{y}(x_m))\right)^2, \quad t \in (0, T) \tag{5}$$
$$N = 1/h, \quad \hat{y}_m = \hat{y}(t_m), \quad x_m = x(t_m), \quad t_m = mh,$$

where the interval $t \in [0,T]$ is divided into N number of steps $t \in (t_0 = 0, t_1 = 0.1, t_2 = 0.2, \ldots, t_N = T)$ with step size h, $\hat{y}(t)$ and $d^2\hat{y}/dt^2$ are FDEN networks given by set of Eq. (2). Similarly, the value ε_2 associated with boundary conditions can be written as:

$$\varepsilon_2 = \frac{1}{2}((y_0 - b)^2 + (y_N - c)^2) \tag{6}$$

It is quite evident that the weights than minimize the objective function ε to approach zero, the approximate solution $\hat{y}(t)$ approaches the exact solution $y(t)$ of the FD equation as given in (1).

Learning methodologies

In this section, a brief introduction as well as parameter setting used for training of weights of FDEN networks, have been presented for SA, PS, GA and ASAs.

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