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A divergence-free low-order stabilized finite element method for a generalized steady state Boussinesq problem

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Abstract

In this work we propose and analyze a new stabilized finite element method for the coupled Navier–Stokes/temperature (or Boussinesq) equations. The method is built using low order conforming elements for velocity and temperature, and piecewise constant elements for pressure. With the help of the lowest order Raviart–Thomas space, a lifting of the jumps of the discrete pressure is built in such a way that when this lifting is added to the conforming velocity field, the resulting velocity is solenoidal (at the price of being non-conforming). This field is then fed to the momentum and temperature equations, guaranteeing that the convective terms in these equations are antisymmetric, without the need of altering them, thus simplifying the analysis of the resulting method. Existence of solutions, discrete stability, and optimal convergence are proved for both the conforming velocity field, and its corresponding divergence-free non-conforming counterpart. Numerical results confirm the theoretical findings, as well as the gain provided by the solenoidal discrete velocity field over the conforming one.

Keywords: Boussinesq problem; Stabilized finite element method; Divergence-free discrete velocity

1. Introduction

In this work we aim at approximating incompressible, non-isothermal flows. We choose as a model problem a generalization of the well-known Boussinesq approximation (see [1,2]). This problem, of high relevance in applications, has been the topic of a number of studies over the last few decades, and a vast number of works have been written about the solvability and approximation of it, using different approaches. We just mention in this, non-extensive, list of references the works [3–13], and the references therein, as examples of the variety of approaches used for tackling this problem.

The particular model problem we choose is a steady-state generalized Boussinesq equation, where the viscosity and thermal conductivity are supposed to be temperature-dependent. The boundary conditions are homogeneous for

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the velocity, but the boundary datum for the temperature is not. This generalization has been the topic of multiple independent studies. In particular, in [14] the existence of solutions is shown using fixed point arguments (the uniqueness is also shown under an assumption of smallness of data and solutions), and these results rely strongly on two fundamental facts. First, a sufficiently small lifting of the boundary datum for the temperature can be proven to exist, in such a way that certain a priori estimates can be derived. Second, the velocity field is divergence-free, which allows the proof of the well-posedness of the fixed-point mapping, and a priori estimates for the fixed-point iterates.

The first restriction mentioned in the last paragraph, namely, the need for a sufficiently small lifting of the boundary datum for the temperature, can be proven if one of the two following assumptions follow (see [15, Section 4] for an in-deep discussion of this fact). First, it follows in a fairly straightforward way if this boundary datum is supposed to be small enough. Alternatively, since the norm in which the lifting needs to be small enough is only the L^3 norm (and thus does not include any derivatives), then this lifting can be proved to have a small enough norm if the mesh is sufficiently refined near the boundary. In such a case it is enough to build the lifting in the usual way, namely, nodal interpolation in the boundary nodes, and zero in all the interior nodes of the triangulation. This latter approach is the one followed in this work, and in our numerical experimentation this has been proven to be sufficient.

The second point raised earlier, namely, the need for the velocity to be divergence-free, is more delicate. As a matter of fact, one common point to most (if not all) discretizations of the Boussinesq problem is the need to guarantee an appropriate level of local mass conservation, as it has been shown that a poor local mass conservation affects the stability, and ultimately, the overall quality of the numerical results. This is especially true if low order conforming elements are used to approximate the velocity field, as the results in [16,13] show. Then, several alternatives have been proposed. One possibility is to rewrite the convective term in its skew-symmetric form. In this way, even if the discrete velocity is not solenoidal (as it happens if conforming, low-order Lagrangian elements are used for velocity, as we use in this work), the convective term remains antisymmetric, thus not affecting stability (see, e.g., the recent works [17] and [10] for the use of this idea). Another possibility is to use divergence-free elements, such as the ones reviewed in the recent paper [18], where some preliminary results are also shown for a natural convection problem. Nevertheless, the applicability of these pairs to more challenging situations, and their stabilization to treat convection-dominated problems, is still to be carried out. A further alternative to produce exactly divergence-free finite element methods is to use BDM pairs of finite elements. This is the approach that has been followed in the fairly recent paper [15], where the analysis presented in [14] has been extended to cover the discrete setting. Nevertheless, some dG-like terms needed to be added to the formulation to enforce its stability, and also the number of degrees of freedom of BDM pairs is higher than for conforming methods.

In this work we propose a method using low order conforming finite element spaces. More precisely, we seek piecewise linear conforming velocities and temperatures, and piecewise constant pressures. Since this choice does not satisfy the inf–sup condition, the pressure is stabilized by penalizing inter-element jumps. It is well-known that the conforming piecewise linear velocity cannot be solenoidal. Thus, we follow the idea presented in [19] where the internal jumps of the discrete pressure are used as degrees of freedom to build a unique element of the lowest order Raviart–Thomas space. This Raviart–Thomas field is then added to the conforming discrete velocity, and the enriched velocity thus built can be proved to be divergence-free. The price to pay for this is the loss of conformity, since this modified velocity field is only div-conforming, but this does not affect the analysis, since the only place in which this modification is used is in the convective terms, where only div-conformity is needed. In [19] this process was presented as a post-processing aimed at recovering a solenoidal discrete velocity at virtually zero cost, producing a discrete velocity that shared some advantages of a non-conforming element at the cost of a conforming one (i.e., needing fewer degrees of freedom). In this work this modification is at the core of the discrete method, since this modified (solenoidal) velocity field is the one fed to the momentum and temperature equations, thus avoiding any modifications of the convective terms. One final important property of this modified velocity field is that it can be proved to enjoy the same convergence properties as the conforming one, in a broken H^1-type norm. For this, we have modified and extended the arguments given in [19], where this fact had been proven for a particular case.

The rest of the manuscript is organized as follows. In Section 2 we present the main notations and preliminary results used throughout. The model problem is presented in Section 3, where we review the main properties of the forms involved in the formulation, and the main results about existence of solutions. The finite element method proposed in this work is presented in Section 4, where we also show the existence of solutions under the appropriate smallness of the data or mesh fine enough near the boundary mentioned previously. The convergence of the method, and a priori error estimates, are proven in Section 5. For the existence and error analyses, we follow closely the general