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# A new family of projection schemes for the incompressible Navier–Stokes equations with control of high-frequency damping

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## Highlights

- Presentation of discrete model problem for assessment of projection schemes including numerical damping.
- Assertion: 2nd order accurate projection schemes do not possess high frequency damping.
- Presentation of two new methodologies based on midpoint rule and generalised-alpha method.
- The proposed schemes offer a compromise between accuracy and high-frequency damping.

## Abstract

A simple spatially discrete model problem consisting of mass points and dash-pots is presented which allows for the assessment of the properties of different projection schemes for the solution of the incompressible Navier–Stokes equations. In particular, the temporal accuracy, the stability and the numerical damping are investigated. The present study suggests that it is not possible to formulate a second order accurate projection/pressure-correction scheme which possesses any high-frequency damping. Motivated by this observation two new families of projection schemes are proposed which are developed from the generalised midpoint rule and from the generalised-$\alpha$ method, respectively, and offer control over high-frequency damping. Both schemes are investigated in detail on the basis of the model problem and subsequently implemented in the context of a finite element formulation for the incompressible Navier–Stokes equations. Comprehensive numerical studies of the flow in a lid-driven cavity and the flow around a cylinder are presented. The observations made are in agreement with the conclusions drawn from the model problem.


*Keywords:* Incompressible Navier–Stokes equations; Fractional step method; Projection method; Finite element method; Generalised-alpha method

## 1. Introduction

In the simulation of incompressible fluid flow, one of the main challenges is posed by the coupling of the velocity and pressure fields through the incompressibility constraint. This has motivated the development of fractional step or

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splitting methods. These methods are based on the decoupling of the velocity–pressure system by splitting it into a sequence of "fractional" or "segregated" solution steps. Although the general idea remains the same, this splitting has been formulated in a number of ways over the years; often in the form of projection methods [1,2], pressure or velocity correction methods [3–5], consistent splitting methods [6], viscosity splitting methods [7] or characteristic-based split (CBS) methods [8], to name just a few. Arguably the most widely used fractional step methods for incompressible fluid flow are the original projection schemes proposed independently by Chorin [1] and Temam [2,9] in the 1960s. In short, these projection methods are based on an orthogonal projection onto a subspace of solenoidal vector fields, see [2] for a thorough explanation. The basic idea is to acquire an intermediate velocity field (Step 1) by solving the momentum equation without the pressure gradient, i.e. considering only viscous, inertia and convection terms, and subsequently computing the pressure and divergence-free end-of-step velocity (Step 2). The appealing benefits of this approach consist in smaller system matrices, dimensionally uniform solution and right hand side vectors and, importantly, the fact that the pressure is obtained efficiently in Step 2 from solving the Poisson equation. The drawbacks of such strategies include additional complexity in the application of the boundary conditions and most of all the introduction of a so-called splitting error, which brings about a relative loss of temporal accuracy compared to a respective coupled approach. Due to their semi-explicit nature, it is crucial that temporal stability and accuracy are in the focus of all development in the area of the fractional step solution schemes.

As mentioned above, in the classical projection methods by Chorin and Temam, the intermediate velocity is computed independently of the pressure. It is well-understood that this restricts these methods to first order accuracy in time. If, in the first step, the pressure is approximated by the solution from the previous time step, then a pressure increment can be computed in the second step and an overall second order accurate scheme can be formulated. This approach is typically known as the "incremental projection" or "pressure correction" method and was first considered in, for instance, [3,4]. It is clear that the accuracy of the pressure extrapolation used in the first step must be increased in order to formulate a more accurate methodology. It is noted that, despite these efforts, the first order accurate schemes are still widely used. The analysis of the properties of the different schemes is not trivial and is an active area of research, see for instance [10–13]. The present work has multiple objectives:

1. Presentation of a discrete model problem consisting of point masses and dash-pots which allows for detailed insight into the properties of projection schemes and is a useful tool for new development;
2. Discussion of high-frequency damping of projection schemes;
3. Presentation of two new families of projection schemes based on the generalised midpoint rule and the generalised-$\alpha$ method [14].

Prior to the further explanation of the objectives, it is pointed out that the work presented in this article is relevant for projection methods based on the finite volume as well as finite element formulations, even though Sections 3 and 4 are set in the context of the finite element method.

Objective 1 is motivated by the successful recent employment of the basic model problems in the area of the partitioned schemes for fluid–structure interaction. Here, the analyses of appropriate spatially discrete model problems has allowed for in-depth insight into temporal and added mass related instabilities [15–17] and is increasingly used for new method development [18,19]. The investigation undertaken in the context of Objective 2 led to the observation that it is impossible to formulate a projection scheme for the model problem which is second order accurate and possesses high-frequency damping. This is an important finding which, to the best of our knowledge, has not been reported elsewhere and which may explain why second order accurate projection schemes have generally not replaced first order schemes. Objective 3 is the attempt to formulate a methodology which is more accurate than basic backward Euler based projection schemes, but offers some high-frequency damping.

The beneficial role of high-frequency damping in incremental numerical solution schemes for partial differential equations in time and space is well-known: The numerical analyst chooses the spatial and temporal discretisation suitable for the length and time scales which are of interest and represent the main system response. Hence, a robust methodology requires high-frequency damping to damp out the effect of the unresolved scales. In particular, high-frequency damping allows for a larger degree of independence between the spatial and temporal resolutions, i.e. a larger range of Courant numbers. In the context of the monolithic solution schemes for computational fluid dynamics, the generalised-$\alpha$ method, which is unconditionally stable, second order accurate and offers control over high-frequency damping, has therefore become very popular, see for instance [20–23]. It was proposed in [14] and is related to its counterpart formulated earlier for solid dynamics in [24] (see also [25]). In the present work, a projection