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# Constraint Energy Minimizing Generalized Multiscale Finite Element Method

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**Abstract**

In this paper, we propose Constraint Energy Minimizing Generalized Multiscale Finite Element Method (CEM-GMsFEM). The main goal of this paper is to design multiscale basis functions within GMsFEM framework such that the convergence of method is independent of the contrast and linearly decreases with respect to mesh size if oversampling size is appropriately chosen. We would like to show a mesh-dependent convergence with a minimal number of basis functions. Our construction starts with an auxiliary multiscale space by solving local spectral problems. In auxiliary multiscale space, we select the basis functions that correspond to small (contrast-dependent) eigenvalues. These basis functions represent the channels (high-contrast features that connect the boundaries of the coarse block). Using the auxiliary space, we propose a constraint energy minimization to construct multiscale spaces. The minimization is performed in the oversampling domain, which is larger than the target coarse block. The constraints allow handling non-decaying components of the local minimizers. If the auxiliary space is correctly chosen, we show that the convergence rate is independent of the contrast (because the basis representing the channels are included in the auxiliary space) and is proportional to the coarse-mesh size (because the constraints handle non-decaying components of the local minimizers). The oversampling size weakly depends on the contrast as our analysis shows. The convergence theorem requires that channels are not aligned with the coarse edges, which hold in many applications, where the channels are oblique with respect to the coarse-mesh geometry. The numerical results confirm our theoretical results. In particular, we show that if the oversampling domain size is not sufficiently large, the errors are large. To remove the contrast-dependence of the oversampling size, we propose a modified construction for basis functions and present numerical results and the analysis.



*Keywords:* Generalized multiscale finite element method; High contrast flow; Local basis functions

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## 1. Introduction

Many practical applications contain multiple scales and high contrast. These include flows in fractured media, processes in channelized porous media and so on. Due to scale disparity and the contrast, some type of coarse-grid models are used to solve these problems. The coarse grid is typically much larger than the fine-grid size and it (the coarse grid) contains many heterogeneities and high contrast. In modeling and simulations of multiscale problems, it is difficult to adjust coarse-grid sizes based on scales and contrast. Thus, it is important that the numerical performance is independent of these physical parameters.

There have been many existing approaches in the literature to handle multiscale problems. In this paper, we focus on Darcy flow equation in heterogeneous media. These multiscale approaches include homogenization approaches [1–3], numerical upscaling methods [4–7], multiscale finite element methods [8], variational multiscale methods [9–13], heterogeneous multiscale methods [14–18], mortar multiscale methods [19–22], localized orthogonal decomposition methods [23], equation-free approaches [24–27], generalized multiscale finite element methods [28–30] and so on. Some of these approaches are based on homogenization methods and compute effective properties. Once the effective properties are computed, the global problem is solved on the coarse grid. Our methods are in the class of multiscale finite element methods, where we seek multiscale basis functions to represent the local heterogeneities. In multiscale methods, one constructs multiscale basis functions that can capture the local oscillatory behavior of the solution.

Our approaches are based on Generalized Multiscale Finite Element Method (GMsFEM), [28–30]. This approach, as MsFEM, constructs multiscale basis functions in each coarse element via local spectral problems. Once local snapshot space is constructed, the main idea of the GMsFEM is to solve local spectral problems and identify multiscale basis functions. These approaches share some common elements with multi-continuum approaches and try to identify high-contrast features that need to be represented individually. These non-local features are typically channels (high-contrast regions that connect the boundaries of the coarse grid) and need separate (individual) basis functions. These observations about representing channels separately are consistent with multi-continuum methods [31]; however, GMsFEM provides a general framework for deriving coarse-grid equations. We note that the localizations of channels are not possible, in general, and this is the reason for constructing basis functions for channels separately as discussed in [32,33]. These ideas are first used in designing optimal preconditioners [33] using overlapping domain decomposition methods, and then in [34,35] using non-overlapping domain decomposition methods. In GMsFEM, the local spectral problems and snapshots, if identified appropriately, correctly identify the necessary channels without any geometric interpretation.

It was shown that the GMsFEM's convergence depends on the eigenvalue decay [36]. However, it is difficult to show a coarse mesh dependent convergence without using oversampling and many basis functions. In this paper, we would like to show a mesh-dependent convergence with a minimal number of basis functions. The convergence analysis of the GMsFEM suggests that one needs to include eigenvectors corresponding to small eigenvalues in the local spectral decomposition. We note that these small eigenvalues represent the channelized features, as we discussed above. To obtain a mesh-dependent convergence, we use the ideas from [23,37–39],[1] which consists of using oversampling domains and obtaining decaying local solutions. Some of these approaches [23] use variational multiscale method framework [9–13,40] to construct subgrid information (see also [41] for the use of multiscale spaces in variational multiscale methods). For high-contrast problems, the local solutions do not decay in channels and thus, we need approaches that can take into account the information in the channels when constructing the decaying local solutions.

The proposed approach starts with auxiliary multiscale basis functions constructed using the GMsFEM in each coarse block. This auxiliary space contains the information related to channels and the number of these basis functions is the same as the number of the channels, which is a minimal number of basis functions required representing high-contrast features. This auxiliary space is used to take care of the non-decaying component of the oversampled local solutions, which occurs in the channels. The construction of multiscale basis functions is done by seeking a minimization of a functional subject to a constraint such that the minimizer is orthogonal (in a certain sense) to the auxiliary space. This allows handling non-decaying component of the oversampled local solutions. The resulting approach contains several basis functions per element and one can use an adaptivity [36,42] to define the basis

[1] We learned about [39] in IPAM workshop (April 2017), which is similar to Section 3 (and Section 5) and done independently and earlier by Tom Hou and Pengchuan Zhang.