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A hybrid a posteriori error estimator for conforming finite element approximations☆

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Highlights

- The hybrid estimator resolves unreliability of ZZ type estimators on coarse meshes.
- The hybrid estimator extends the improved ZZ estimator to higher order elements.
- The hybrid estimator is explicit and is more accurate than the residual estimator.

Abstract

This paper introduces a hybrid a posteriori error estimator for the conforming finite element method, which may be regarded as a combination of the explicit residual and the improved ZZ error estimators. With comparable cost, the hybrid estimator is more accurate than the residual estimator. It is shown that the hybrid estimator is reliable on all meshes, unlike estimators of the ZZ type. Moreover, the reliability constant is independent of the jump of the diffusion coefficients for elliptic interface problems under the monotonicity assumption of the coefficients. Finally, numerical examples confirm the robustness of the estimator with respect to coefficient jumps and also better effectivity index compared to the residual estimator.

Keywords: Finite element method; A posteriori error estimation; Adaptive mesh refinement; Diffusion problem

1. Introduction

Adaptive mesh refinement is necessary in the discretization of partial differential equations (PDEs) in order to handle computational challenges [1]. A posteriori error estimates play a crucial role in adaptive mesh refinement, where one tries to estimate the error by computing quantities (called error estimators) based on numerical solution as well as data from the underlying PDE. It is well known that the explicit residual error estimators (see, e.g., [2–6]) are computationally inexpensive with applications to a large class of problems. Moreover, for computationally challenging problems such as interface problems, proper weighted residual estimators (see, e.g., [5,7]) generate efficient meshes.

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However, it is also known that residual estimators usually overestimate the true error by a large margin compared to estimators of the Zienkiewicz–Zhu (ZZ) type (cf. [8]). In this paper we introduce a hybrid a posteriori error estimator for the conforming finite element method, which is more accurate than the residual error estimator and is reliable on all meshes unlike the ZZ estimators.

By first recovering a gradient in the conforming C^0 linear vector finite element space from the numerical gradient, the Zienkiewicz–Zhu (ZZ) estimator [9] is defined as the L^2 norm of the difference between the recovered and the numerical gradients. Due to its simplicity, universality, and asymptotic exactness for smooth problems, the ZZ estimator enjoys a high popularity in the engineering community (see, e.g., [6,10–12]).

Despite its popularity, it is also well known that estimators of the ZZ type have several major drawbacks. First, adaptive mesh refinement (AMR) algorithms using the ZZ estimator are not efficient to reduce global error for non-smooth problems, e.g., interface problems (see, e.g., [13]). By exploring the mathematical structure of the underlying problem and the characteristics of finite element approximations, [14] identified the reason for this failure, and [15] introduced an improved ZZ estimator for the lowest order conforming elements which is explicit and efficient for non-smooth problems. Second, estimators of the ZZ type are not reliable on coarse meshes relative to the underlying problem. A simple one-dimensional example in [4] shows that the ZZ estimator equals zero but the true error is arbitrarily large. For a two-dimensional example, see Section 5.4. Moreover, estimators of the ZZ type work only for elements of the lowest order and research for higher order elements is still in its infancy (see, e.g., [16,17]). Bank, Xu, and Zheng in [16] recovered higher order derivatives, and Naga and Zhang in [17] approximated the numerical solution by a higher order polynomial. Both the approaches demonstrate some appealing features like super-convergence and asymptotic exactness under certain smoothness assumptions of the exact solution.

Comparing with the residual estimator, we realized that the scaled element residual is no longer higher order when the recovered flux is the L^2 projection of the numerical flux in an $H(\text{div})$-conforming space. By simply adding an appropriately weighted element residual to the improved ZZ estimator, it was shown in [18] that the resulting estimator for higher order elements is reliable on all meshes and more accurate than the residual estimator. Computing the L^2 projection of the numerical flux in an $H(\text{div})$-conforming space requires solving a global problem and, hence, the estimator in [18] is more expensive than the residual estimator.

The purpose of this paper is to introduce an explicit flux recovery in an $H(\text{div})$-conforming space so that the resulting hybrid error estimator is more accurate than the residual estimator with similar computational cost and applicability. To do so, we first specify the desired normal component of the recovered flux on each face as a weighted average of the normal components of the numerical fluxes. Then the recovered flux is chosen to satisfy a compatible divergence equation on each element. In particular, we are able to derive an explicit formula for a recovered flux in an $H(\text{div})$-conforming space and the formula is automatically valid for higher order finite element approximations. Unlike existing ZZ-type estimators, which are not reliable on coarse meshes, we incorporate the divergence error in the estimator and the resulting error estimator of hybrid type is proved to be reliable on all meshes.

This hybrid estimator displays a strong connection to the explicit residual estimator as we can prove that the proposed estimator is actually equivalent to the residual estimator [5] with constants independent of the diffusion coefficients (see Section 4.2). As a result, the robustness of the residual estimator with respect to coefficient jumps carries over to the hybrid estimator. Despite the theoretical equivalence, numerical results show that the hybrid estimator is more accurate than the residual estimator. Hence the hybrid estimator can be viewed as a substitute of the residual estimator with an improved accuracy. The innate link to the residual estimator lends comparable generality to the hybrid estimator and future work includes applying the technique to convection–diffusion problems.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem with a conforming finite element discretization and some notation. In Section 3, we present the explicit flux recovery. In Section 4, after defining the element indicator and the resulting global error estimator, we prove the robust local efficiency and global reliability. The equivalence between the proposed local indicator and the standard residual-based indicator is established in Section 4.2. Numerical results are presented in Section 5 to demonstrate the performance of the proposed estimator and a counter example is included in the end to illustrate the unreliability of ZZ-type estimators on a coarse mesh.

2. Problem and finite element approximation

Let Ω be a bounded polygonal domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega$, where $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, assume that $\text{meas}(\Gamma_D) > 0$. Consider the diffusion equation

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega \tag{2.1}$$