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# Observer-based feedback boundary stabilization of the Navier–Stokes equations

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**Abstract**

This paper aims at designing an observer-based feedback law which locally stabilizes the solution to the two dimensional Navier–Stokes equations with mixed boundary conditions. We consider a finite number of controls acting on a portion of the boundary through Robin boundary conditions and construct a linear Luenberger observer based on the point observations of the linearized Navier–Stokes equations. The sensor location for the point observations is determined by the response of feedback functional gains. We prove that the nonlinear system coupled with the observer through the feedback law is locally exponentially stable. Numerical experiments based on a Taylor–Hood finite element method are presented to illustrate the design for different Reynolds numbers.


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*Keywords:* Navier–Stokes equations; Mixed boundary conditions; Feedback stabilization; Point observation; Luenberger observer

## 1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded and connected domain with Lipschitz boundary $\Gamma$. Consider the Navier–Stokes equations

$$\frac{\partial v}{\partial t} - \nu \operatorname{div}(\nabla v + (\nabla v)^T) + (v \cdot \nabla)v + \nabla p = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

with initial condition $v(0) = v_0$, where $v$ is the velocity, $p$ is the pressure, $\nu$ is the viscosity (or the inverse of the Reynolds number, i.e., $\nu = \frac{1}{Re}$), and $f$ is a time independent external body force. It is well known that a flow can transit from a laminar to turbulent state when the Reynolds number reaches a certain critical value or small

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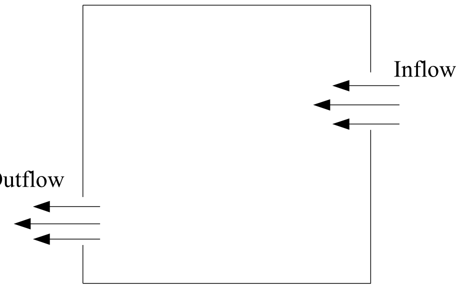


**Fig. 1.** Domain $\Omega$.

disturbances are introduced to the flow (cf. [1–7]). In this paper, we are interested in stabilizing equations (1.1)–(1.2) in a neighborhood of a possible unstable steady-state solution based on partial estimation of the system. The controls act on a portion of the boundary $\Gamma$. The problem of feedback boundary stabilization for the Navier–Stokes equations (1.1)–(1.2) has been widely studied (cf. [8–15]). By using Linear Quadratic Regulator (LQR) control design, Barbu, Lasiecka and Triggiani considered the Dirichlet boundary control with normal component zero in [10,11]. Later on, Raymond studied the case with normal component nonzero in [13,14]. Moreover, Badra and Takahashi in [9] and Raymond and Thevenet in [15] showed that Dirichlet boundary control can be finite dimensional. In [8], Badra also studied the Neumann boundary control. However, these results considered either only Dirichlet or Neumann boundary conditions on the entire boundary. Recently, Raymond and Nguyen in [12] discussed the case with mixed boundary conditions, where a localized Dirichlet boundary control is applied. As mentioned in [12], one of the difficulties arising in the mixed boundary case is that a singularity occurs at junctions between Dirichlet and Neumann boundary conditions, therefore, the solution to the stationary Navier–Stokes is less regular than the case with only Dirichlet or Neumann boundary conditions. Furthermore, Dirichlet boundary control with normal component nonzero results in a rather complicated Riccati-like equation, which is difficult to solve by standard numerical methods (cf. [16–18]).

Our current work is concerned with a two dimensional problem with mixed boundary conditions, where the control input is Neumann/Robin type and finite dimensional. Homogeneous Dirichlet boundary condition is imposed on the rest of the boundary. This is motivated by the problems of designing and controlling energy efficient buildings, where, for example, the vents act as artificial boundaries for the controlled airflow to enter and exit. Under this setting, a standard Riccati equation can be established. Also, under suitable conditions Robin boundary conditions can be used to approximate Dirichlet boundary conditions (cf. [19–21]). However, the feedback law based on LQR design requires full state information. This assumes that the whole state of the system can be measured, which is often not practical for the flow field. A more practical situation occurs when there is only partial information of the flow accessible. The goal of this paper is to construct a feedback law based on the partial information measured from the linearized system, which stabilizes the full nonlinear system. In particular, we employ point observations for the output measurement and then construct a linear Luenberger observer, where we also address the issue of optimal sensor locations. In the end, a Taylor–Hood finite element method is employed to implement the numerical simulations.

Challenges arise in the present work due to mixed boundary conditions, where singularities may occur at the junctions of different types of boundary conditions, and hence the solution to (1.1)–(1.2) is not smooth in general, no matter how regular the data are. One of the main objectives of the present work is to tackle the situation where the linearized system has very low regularity. In order to focus on the ideas, consider that the boundary $\Gamma = \overline{\Gamma}_{\mathrm{I}} \cup \overline{\Gamma}_{\mathrm{O}} \cup \overline{\Gamma}_{\mathrm{D}}$, where $\Gamma_{\mathrm{I}}$, $\Gamma_{\mathrm{O}}$, and $\Gamma_{\mathrm{D}}$ are open and mutually disjoint subsets of $\Gamma$. A Robin type inflow control is introduced through the inlet $\Gamma_{\mathrm{I}}$, a stress free condition is prescribed on the outlet $\Gamma_{\mathrm{O}}$, and a no-slip boundary condition is imposed on the rest of the boundary $\Gamma_{\mathrm{D}}$. In the case of polyhedral domains, we assume that the angles at the edges where the boundary conditions change are less than or equal to $\pi$. In the present work, we are particularly interested in the case that $\Gamma_{\mathrm{I}}$ and $\Gamma_{\mathrm{O}}$ meet with $\Gamma_{\mathrm{D}}$ tangentially as shown in Fig. 1. This simple zone configuration is typical of the systems of interest and will be used to illustrate the theoretical and numerical results developed below.

Let $\mathcal{T}(v, p)$ denote the Cauchy stress tensor

$$\mathcal{T}(v, p) = 2\nu\epsilon(v) - pI,$$